Chapter 9

More About Graphs

Discrete Mathematics I on 7 May 2012

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   Floyd-Warshall Algorithm
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Huynh Tuong Nguyen, Tran Vinh Tan

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Paths and Circuits

Simple path of length 4

Circuit of length 4
Path and Circuits

Definition (in undirected graph)

- **Path (đường đi)** of length $n$ from $u$ to $v$: a sequence of $n$ edges $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.

- A path is a **circuit (chu trình)** if it begins and ends at the same vertex, $u = v$.

- A path or circuit is **simple (đơn)** if it does not contain the same edge more than once.
Path and Circuits

Definition (in directed graphs)

Path is a sequence of \((x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n)\), where \(x_0 = u\) and \(x_n = v\).
Connectedness in Undirected Graphs

Definition

- An undirected graph is called connected (liên thông) if there is a path between every pair of distinct vertices of the graph.
- There is a simple path between every pair of distinct vertices of a connected undirected graph.

Connected graph
Disconnected graph
Connected components (thành phần liên thông)
How Connected is a Graph?

Definition

- \( b \) is a cut vertex (đỉnh cắt) or articulation point (điểm khớp).
- \( \{a, b\} \) is a cut edge (cạnh cắt) or bridge (cầu). What else?

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How Connected is a Graph?

**Definition**

- This graph don't have cut vertices: nonseparable graph (đồ thị không thể phân tách)
- The vertex cut is \{c, f\}, so the minimum number of vertices in a vertex cut, vertex connectivity (liên thông đỉnh) \(\kappa(G) = 2\).
- The edge cut is \{\{b, c\}, \{a, f\}, \{f, g\}\}, the minimum number of edges in an edge cut, edge connectivity (liên thông cạnh) \(\lambda(G) = 3\).
Applications of Vertex and Edge Connectivity

- Reliability of networks
  - Minimum number of routers that disconnect the network
  - Minimum number of fiber optic links that can be down to disconnect the network
- Highway network
  - Minimum number of intersections that can be closed
  - Minimum number of roads that can be closed
Connectedness in Directed Graphs

**Definition**

- An directed graph is **strongly connected** *(liên thông mạnh)* if there is a path between any two vertices in the graph (for both directions).
- An directed graph is **weakly connected** *(liên thông yếu)* if there is a path between any two vertices in the underlying undirected graph.

![Directed Graphs Diagram](https://fb.com/tailieudientucntt)

**Strongly connected**

- $a$ to $b$
- $b$ to $a$
- $a$ to $c$
- $c$ to $a$
- $c$ to $d$
- $d$ to $c$
- $d$ to $e$
- $e$ to $d$

**Weakly connected**

- $a$ to $b$
- $b$ to $a$
- $a$ to $c$
- $c$ to $a$
- $c$ to $d$
- $d$ to $c$
- $d$ to $e$
- $e$ to $d$

Applications

Example

Determine whether the graphs below are isomorphic.

\[ G \]
\[ H \]

Solution

\[ H \] has a simple circuit of length three, not \[ G \].
Applications

Example

Determine whether the graphs below are isomorphic.

Solution

Both graphs have the same vertices, edges, degrees, circuits. They **may** be isomorphic.

To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degrees.
The Famous Problem of Seven Bridges of Königsberg

- Is there a route that a person crosses all the seven bridges once?
Euler Solution

- Euler gave the solution: It is **not** possible to cross all the bridges exactly once.
What is Euler Path and Circuit?

- **Euler Path** (đường đi Euler) is a path in the graph that passes each edge only once. The problem of Seven Bridges of Königsberg can be also stated: Does Euler Path exist in the graph?

- **Euler Circuit** (chu trình Euler) is a path in the graph that passes each edge only once and return back to its original position. From Definition, Euler Circuit is a subset of Euler Path.
Examples of Euler Path and Circuit

Euler Circuit

Euler Path

Examples of Euler Path and Circuit

Euler Circuit

Euler Path
Conditions for Existence

In a connected multigraph,

- Euler Circuit existence: **no odd-degree nodes exist** in the graph.
- Euler Path existence: **2 or no odd-degree nodes exist** in the graph.
Back to the Seven Bridges Problem

- Four vertices of odd degree
- No Euler circuit → cannot cross each bridge exactly once, and return to starting point
- No Euler path, either
Searching Euler Circuits and Paths – Fleury’s Algorithm

- Choose a random vertex (if circuit) or an odd degree vertex (if path)
- Pick an edge joined to another vertex so that it is not a cut edge unless there is no alternative
- Remove the chosen edge. The above procedure is repeated until all edges are covered.
Searching Euler Circuits and Paths – Hierholzer’s Algorithm

- Choose a starting vertex and find a circuit
- As long as there exists a vertex \( v \) that belongs to the current tour but that has adjacent edges not part of the tour, start another circuit from \( v \)

More efficient algorithm, \( O(n) \).
Exercise

Example

Are these following graph Euler path (circuit)? If yes, find one.
Traveling Salesman Problem

Is there the possible tour that visits each city exactly once?
What Is A Hamilton Circuit?

Definition

The circuit that visit each vertex in a graph once
Rules of Hamilton Circuits

\[ \text{deg}(v) = 2 \text{ for } \forall v \text{ in Hamilton circuit!} \]

**Rule 1** if \( \text{deg}(v) = 2 \), both edge must be used.

**Rule 2** No subcircuit (\( chu \text{ trinh con} \)) can be formed.

**Rule 3** Once two edges at a vertex \( v \) is determined, all other edges incident at \( v \) must be removed.
Finding Hamilton Circuits

Vertices: cities
Edges: possible routes

Rule 1
\[ \text{deg}(v) = 2 \]

Rule 3
Once two edges are determined, other edges must be removed

We get Hamilton circuit!
Existence of Hamilton Circuit

Hamilton circuit does not exist for all graph. But, there is no specific way to find whether Hamilton circuit exists or not.

Simple check by rules of Hamilton circuit

Violates Rule 2! (No subcircuit)
We can verify nonexistence of the graph during find Hamilton circuit.

Contradict with Rule 1!

Hamilton circuit doesn’t exist!
Application – Gray Code

Definition

The binary sequence that express consecutive numbers by differing just one position of sequence.

<table>
<thead>
<tr>
<th>Decimal number</th>
<th>Binary number</th>
<th>Gray code</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>= 001</td>
<td>= 000</td>
</tr>
<tr>
<td>2</td>
<td>= 010</td>
<td>= 100</td>
</tr>
<tr>
<td>3</td>
<td>= 011</td>
<td>= 110</td>
</tr>
<tr>
<td>4</td>
<td>= 100</td>
<td>= 010</td>
</tr>
<tr>
<td>5</td>
<td>= 101</td>
<td>= 011</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Used at digital communication for reduce the effect of noise; it prevents serious changes of information by noise.
Gray Code

An \(n\)-digit gray code can be generated by finding Hamilton circuits of an \(n\)-dimensional hypercube. Consider the case \(n = 3\).

```
5 = 011
6 = 111
8 = 001
3 = 101
6 = 010
5 = 110
1 = 000
2 = 100
```

Coordinate of each vertex is 3-digit binary sequences. Coordinates of adjacent vertices differ in just one place. Hamilton circuits of a cubic graph makes the order of binary sequences!
Weighted Graphs
The problem is also sometimes called the single-pair shortest path problem, to distinguish it from the following generalizations:

- The **single-source shortest path problem**, in which we have to find shortest paths from a source vertex \( v \) to all other vertices in the graph.

- The **single-destination shortest path problem**, in which we have to find shortest paths from all vertices in the graph to a single destination vertex \( v \). This can be reduced to the single-source shortest path problem by reversing the edges in the graph.

- The **all-pairs shortest path problem**, in which we have to find shortest paths between every pair of vertices \( v, v' \) in the graph.

These generalizations have significantly more efficient algorithms than the simplistic approach of running a single-pair shortest path algorithm on all relevant pairs of vertices.
Dijkstra's Algorithm

procedure Dijkstra(G,a)
  // Initialization Step
 forall vertices v
    Label[v] := ∞
    Prev[v] := -1
  endfor
  Label(a) := 0 // a is the source node
  S := ∅

  // Iteration Step
  while z ∉ S
    u := a vertex not in S with minimal Label
    S := S ∪ {u}
    forall vertices v not in S
      if (Label[u] + Wt(u,v)) < Label(v)
        then begin
          Label[v] := Label[u] + Wt(u,v)
          Pred[v] := u
        end
  endwhile
Example

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>z</th>
</tr>
</thead>
<tbody>
<tr>
<td>∅</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>a</td>
<td>0</td>
<td>4</td>
<td>2</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>c</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>10</td>
<td>12</td>
<td>∞</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>12</td>
<td>∞</td>
</tr>
<tr>
<td>d</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>10</td>
<td>14</td>
</tr>
<tr>
<td>e</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>8</td>
<td>10</td>
<td>13</td>
</tr>
</tbody>
</table>
Example

\[\begin{array}{ccccccc}
S & a & b & c & d & e & z \\
\emptyset & 0 & \infty & \infty & \infty & \infty & \infty \\
a & 0 & 4 & 22 & \infty & \infty & \infty \\
c & 0 & 33 & 2 & 10 & 12 & \infty \\
b & 0 & 3 & 2 & 8 & 12 & \infty \\
d & 0 & 3 & 2 & 8 & 10 & 14 \\
e & 0 & 3 & 2 & 8 & 10 & 13 \end{array}\]
Back tracking procedure

How to determine shortest path from $a$ to $d$ according to Dijkstra’s algorithm?
Dijkstra’s Algorithm

**Property**
Applicable for any $G$, any length $\ell(v_i) \geq 0$, $\forall i$; one-to-all; complexity $O(|V|^2)$. 
Exercise

Example

Find the shortest path from $e$ to other vertices using Dijkstra’s algorithm.
Dijkstra’s Algorithm Flaw

Can Dijkstra’s Algorithm be used on...

- ...digraph?
  - Yes!
- ...negative weighted graph?
  - No! Why?
**Bellman-Ford Algorithm**

```plaintext
procedure BellmanFord(G, a)
    // Initialization Step
    forall vertices v
    Label[v] := \infty
    Prev[v] := -1
    Label(a) := 0  // a is the source node

    // Iteration Step
    for i from 1 to size(vertices)-1
        forall vertices v
        if (Label[u] + Wt(u,v)) < Label[v]
            then
                Label[v] := Label[u] + Wt(u,v)
                Prev[v] := u

    // Check circuit of negative weight
    forall vertices v
    if (Label[u] + Wt(u,v)) < Label(v)
        error "Contains circuit of negative weight"
```

**Property**

any $G$, any length; one-to-all; detect whether there exists a circle of negative length; complexity $\mathcal{O}(|V| \times |E|)$.
## Example

<table>
<thead>
<tr>
<th>Step</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-2a$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$\infty$</td>
<td>$3a$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$-2$</td>
<td>$3b$</td>
<td>$\infty$</td>
<td>$-5b$</td>
<td>$3$</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>$-2$</td>
<td>$3$</td>
<td>$-4e$</td>
<td>$-5$</td>
<td>$3$</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>$-2$</td>
<td>$3$</td>
<td>$-4$</td>
<td>$-5$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

Stop since Step 4 = Step 3.
Backtracking procedure

Example

<table>
<thead>
<tr>
<th>Step</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-2a</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>3a</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>3b</td>
<td>∞</td>
<td>-5b</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-2</td>
<td>3</td>
<td>-4e</td>
<td>-5</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>-2</td>
<td>3</td>
<td>-4</td>
<td>-5</td>
<td>3</td>
</tr>
</tbody>
</table>

Stop since Step 4 = Step 3.

How to find shortest path from a to d? $a \rightarrow b \rightarrow e \rightarrow d$
Example

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<table>
<thead>
<tr>
<th>Step</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-2a</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
<td>∞</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>-1b</td>
<td>∞</td>
<td>-1b</td>
<td>∞</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>-2</td>
<td>-1</td>
<td>-6c</td>
<td>-1</td>
<td>-4c</td>
</tr>
<tr>
<td>4</td>
<td>-1f</td>
<td>-2</td>
<td>-1</td>
<td>-6</td>
<td>-3f</td>
<td>-4</td>
</tr>
<tr>
<td>5</td>
<td>-1</td>
<td>-3a</td>
<td>-1</td>
<td>-6</td>
<td>-3</td>
<td>-4</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td>-3</td>
<td>-2b</td>
<td>-6</td>
<td>-3</td>
<td>-4</td>
</tr>
<tr>
<td>7</td>
<td>-1</td>
<td>-3</td>
<td>-2</td>
<td>-7c</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

There exists a circle of negative length since Step 6 ≠ Step 5.
Exercise

[Graph]

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Exercise

- Euler Paths and Circuits
- Hamilton Paths and Circuits
- Shortest Path Problem
  - Dijkstra's Algorithm
  - Bellman-Ford Algorithm
  - Floyd-Warshall Algorithm
- Ford's algorithm
- Traveling Salesman Problem
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Exercise

```
Exercise
```

```
Exercise
```

```
Exercise
```
Floyd-Warshall Algorithm [1962]

\[
\text{procedure FloydWarshall ()}
\]
\[
\text{for k := 1 to n}
\]
\[
\text{for i := 1 to n}
\]
\[
\text{for j := 1 to n}
\]
\[
\text{path}[i,j] = \min (\text{path}[i,j],
\text{path}[i,k] + \text{path}[k,j]);
\]

**Property**

any \( G \), any length; all-to-all; this is an software algorithm; complexity \( O(|V|^3) \).
Example

Shortest path from $b$ to $d$
(53 from $L^{(4)}$):
$bd = bc + cd$
($53 = -20 + 71$ from $L^{(3)}$)
$cd = ca + ad$
($71 = 30 + 40$ from $L^{(1)}$)
$\Rightarrow bd = bc + ca + ad$

$L^{(0)} = \begin{pmatrix}
0 & 10 & \infty & 40 \\
20 & 00 & -20 & \infty \\
30 & \infty & 00 & \infty \\
\infty & -50 & -10 & 00
\end{pmatrix}$

$L^{(1)} = \begin{pmatrix}
0 & 10 & \infty & 40 \\
20 & 00 & -20 & 61 \\
30 & 41 & 00 & 71 \\
\infty & -50 & -10 & 00
\end{pmatrix}$

$L^{(2)} = \begin{pmatrix}
0 & 10 & -12 & 40 \\
20 & 00 & -20 & 61 \\
3 & 41 & 00 & 71 \\
-32 & -50 & -72 & 00
\end{pmatrix}$

$L^{(3)} = \begin{pmatrix}
13 & 00 & -20 & 53 \\
3 & 41 & 00 & 71 \\
-43 & -50 & -72 & 00 \\
00 & 10 & -34 & 40
\end{pmatrix}$

$L^{(4)} = \begin{pmatrix}
13 & 00 & -20 & 53 \\
30 & 24 & 00 & 71 \\
-43 & -50 & -72 & 00
\end{pmatrix}$
Example

\[ L^{(0)} = \begin{pmatrix} 0 & 2 & \infty \\ 3 & 0 & -4 \\ 1 & \infty & 0 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 0 & 2 & \infty \\ 3 & 0 & -4 \\ 1 & 3 & 0 \end{pmatrix} \]

\[ L^{(2)} = \begin{pmatrix} 0 & 2 & -2 \\ 3 & 0 & -4 \\ 1 & 3 & -1 \end{pmatrix} \]

STOP, there exists a circuit of negative length.

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Ford’s algorithm

\[ \pi(1) = 0 \]

For each \( j \in V \) do

\[ \pi(j) = \min_{i \in \rho_j^{-1}} (\pi(i) + \ell[i,j]) \]

End

Property

\( G \) without circle, positive length; one-to-all; rank table definition; complexity \( O(|V|) \).
Example

![Graph Example]

<table>
<thead>
<tr>
<th>i</th>
<th>$\Gamma_i^{-1}$</th>
<th>rank(i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>A,B</td>
<td>2</td>
</tr>
<tr>
<td>D</td>
<td>B,C,E</td>
<td>3</td>
</tr>
<tr>
<td>E</td>
<td>A</td>
<td>1</td>
</tr>
</tbody>
</table>

Example

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Application

Problem

A young professor in Hue is invited to teach some years in Ho Chi Minh university of technology. He decides to represent the diverse operations of his transfer by a graph and, in this purpose, establishes the list of following operations:

A: Find a house in Ho Chi Minh city.
B: Choose a removal man and sign a contract of move
C: Make pack his furniture by the removal man
D: Make transport his furniture towards Ho Chi Minh city
E: Find an accommodation to HCM (from Hue)
F: Transport his family to HCM
G: Move into his new accommodation
H: Register the children to their new school
I: Look for a temporary work for his wife
J: Fit out the new accommodation and pay this arrangement with the first treatment of his wife
K: Find a small bar to celebrate in family the success of the move and express the enjoyment to live in a good accommodation arrangement
Application

Considering constraint of posteriority following: $A < F; B < C; C < D \land F; D < G; E < F; F < G \land H \land I; G < K; H < K; I < J; J < K$.

Approximated job processing times:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
<th>J</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>7</td>
<td>3</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>2</td>
</tr>
</tbody>
</table>

**Question**

- Determine a schedule of the 'movement' with minimal duration.
- What happens if his new accommodation is not available before date 20? In that case, of what margin we have to make the task $J$?
How to determine a shortest path from $u$ to $v$ in graph $G$ which traverses at most $\leq$ a given constant number of intermediate vertices.
### Problem

- Given a set of $n$ customers located in $n$ cities and distances for each pair of cities, the problem involves finding a round-trip with the minimum traveling cost.

- The vehicle must visit each customer exactly once and return to its point of origin also called depot.

- The objective function is the total cost of the tour.

- $\mathcal{NP}$-complete: all known techniques for obtaining an exact solution require an exponentially increasing number of steps (computing resources) as the problems become larger.

- **TSP is one of the most intensely studied problems in computational mathematics, yet no effective solution method.**
The total number of possible Hamilton circuit is \((n - 1)!/2\).

For example, if there are 25 customers to visit, the total number of solutions is \(24!/2 = 3.1 \times 10^{23}\).

If the depot is located at node 1, then the optimal tour is 1 - 5 - 2 - 3 - 4 - 1 with total cost equal to 11.
Vehicle Routing Problem

Problem

- The vehicle routing problem involves finding a set of trips, one for each vehicle, to deliver known quantities of goods to a set of customers.
- The objective is to minimize the travel costs of all trips combined.
- There may be upper bounds on the total load of each vehicle and the total duration of its trip.
- The most basic Vehicle Routing Problem (VRP) is the single-depot capacitate VRP.
Planar Graphs

-Connectivity
-Paths and Circuits
-Euler and Hamilton
-Paths
-Euler Paths and Circuits
-Hamilton Paths and Circuits
-Shortest Path Problem
-Dijkstra’s Algorithm
-Bellman-Ford Algorithm
-Floyd-Warshall Algorithm
-Ford’s algorithm
-Traveling Salesman Problem

Graph Coloring

More About Graphs
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Tran Vinh Tan

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Planar Graphs

**Definition**

- A graph is called *planar* (*phẳng*) if it can be drawn in the plane *without any edges crossing*.
- Such a drawing is called *planar representation* (*biểu diễn phẳng*) of the graph.

\[ K_4 \quad K_4 \text{ with no crossing} \]
Important Corollaries

Corollary

1. If $G$ is a connected planar simple graph with $e$ edges and $v$ vertices where $v \geq 3$, then $e \leq 3v - 6$.

2. If $G$ is a connected planar simple graph with $e$ edges and $v$ vertices where $v \geq 3$, and no circuits of length 3, then $e \leq 2v - 4$.

$K_{3,3}$
Non-planar

$K_5$
Non-planar
**Definition**

- Given a planar graph $G$, an **elementary subdivision** (phân chia sơ cấp) is removing an edge $\{u, v\}$ and adding a new vertex $w$ together with edges $\{u, w\}$ and $\{w, v\}$.
- Graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called **homeomorphic** (đồng phôi) if they can obtained from the same graph by a sequence of elementary subdivisions.
Kuratowski’s Theorem

Theorem

A graph is nonplanar iff it contains a subgraph homeomorphic to $K_{3,3}$ or $K_5$. 
Exercise

- Is $K_4$ planar?
- Is $Q_3$ planar?
Maps and Graphs

**Definition**

- Every map can be represented by a graph. We call it dual graph.
- Problem of coloring the regions of a map $\rightarrow$ coloring the vertices of the dual graph so that no two adjacent vertices have the same color.

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- $A$ - $B$ - $C$ - $D$ - $E$
Graph coloring

Definition

- A coloring (tô màu) of a simple graph is the assignment of a color to each vertex of the graph so that no two adjacent vertices are assigned the same color.

- The chromatic number (số màu) of a graph, denoted by $\chi(G)$, is the least number of colors needed for a coloring of this graph.
Four color theorem

Theorem (Four color theorem)

_The chromatic number of a planar graph is no greater than four._

- Was a conjecture in the 1850s
- Was not proved completely until 1976 by Kenneth Appel and Wolfgang Haken, using computer
- No proof not relying on a computer has yet been found
Applications of Graph coloring

Scheduling Final Exam

- How can the final exams at a university be scheduled so that no student has two exams at the same time?
- Suppose we have 7 finals, numbered 1 through 7.
- The pairs of courses have common students are depicted in the following graph

![Graph Diagram](https://fb.com/tailiedientucntt)
Applications of Graph Coloring

Other Applications

- **Frequency Assignments**: Television channels 2 through 12 are assigned to stations in North America so that no two stations within 150 miles can operate on the same channel. How can the assignment of channels be modeled by graph coloring?

- **Index Registers**: In an execution of loop, the frequently used variables should be stored in index registers to speed up. How many index registers are needed?