Chapter 4

Functions

Discrete Mathematics I on 13 March 2012

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2. One-to-one and Onto Functions

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4. Recursion
Introduction

Each student is assigned a grade from set \{0, 0.1, 0.2, 0.3, \ldots, 9.9, 10.0\} at the end of semester.
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• Function is extremely important in mathematics and computer science
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  - linear, polynomial, exponential, logarithmic,...
Introduction

• Each student is assigned a grade from set \( \{0, 0.1, 0.2, 0.3, \ldots, 9.9, 10.0\} \) at the end of semester
• Function is extremely important in mathematics and computer science
  • linear, polynomial, exponential, logarithmic,…
• Don’t worry! For discrete mathematics, we need to understand functions at a basic set theoretic level
Function

**Definition**

Let $A$ and $B$ be nonempty sets. A **function** $f$ from $A$ to $B$ is an assignment of **exactly one** element of $B$ to each element of $A$.

- $f: A \rightarrow B$
- $A$: domain (miền xác định) of $f$
- $B$: codomain (miền giá trị) of $f$
- For each $a \in A$, if $f(a) = b$, $b$ is an image (ảnh) of $a$
- $a$ is pre-image (nghịch ảnh) of $f(a)$
- Range of $f$ is the set of all images of elements of $A$
- $f$ maps (ánh xạ) $A$ to $B$

$A$ $B$

$f(a) = b$
Function

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- $a = f(a)$
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- **Range** of $f$ is the set of all images of elements of $A$
- $f$ maps (ánh xạ) $A$ to $B$

![Diagram of function](https://fb.com/tailieudientucntt)
Example

- $y$ is an image of $d$
- $c$ is a pre-image of $z$
Example:

- \( y \) is an image of \( d \)
- \( c \) is a pre-image of \( z \)
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Example

What are domain, codomain, and range of the function that assigns grades to students includes: student A: 5, B: 3.5, C: 9, D: 5.2, E: 4.9?

Example

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ assign the square of an integer to this integer. What is $f(x)$? Domain, codomain, range of $f$?
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- $f(x) = x^2$
- Domain: set of all integers
- Codomain: Set of all integers
- Range of $f$: \{0, 1, 4, 9, \ldots\}
Example

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Let \( f : \mathbb{Z} \to \mathbb{Z} \) assign the the square of an integer to this integer. What is \( f(x) \)? Domain, codomain, range of \( f \)?

- \( f(x) = x^2 \)
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- Codomain: Set of all integers
- Range of \( f \) : \( \{0, 1, 4, 9, \ldots\} \)
Add and multiply real-valued functions

Definition

Let \( f_1 \) and \( f_2 \) be functions from \( A \) to \( \mathbb{R} \). Then \( f_1 + f_2 \) and \( f_1 f_2 \) are also functions from \( A \) to \( \mathbb{R} \) defined by

\[
(f_1 + f_2)(x) = f_1(x) + f_2(x)
\]

\[
(f_1 f_2)(x) = f_1(x) f_2(x)
\]
Add and multiply real-valued functions

### Definition
Let $f_1$ and $f_2$ be functions from $A$ to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from $A$ to $\mathbb{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$

$$(f_1 f_2)(x) = f_1(x)f_2(x)$$

### Example
Let $f_1(x) = x^2$ and $f_2(x) = x - x^2$. What are the functions $f_1 + f_2$ and $f_1 f_2$?

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$$

$$(f_1 f_2)(x) = f_1(x)f_2(x) = x^2(x - x^2) = x^3 - x^4$$
Image of a subset

Definition

Let $f : A \rightarrow B$ and $S \subseteq A$. The image of $S$:

$$f(S) = \{f(s) \mid s \in S\}$$
**Image of a subset**

**Definition**

Let \( f : A \rightarrow B \) and \( S \subseteq A \). The image of \( S \):

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Let \( f : A \rightarrow B \) and \( S \subseteq A \). The image of \( S \):

\[
 f(S) = \{ f(s) \mid s \in S \}
\]

\[
 f(\{a, b, c, d\}) = \{x, y, z\}
\]
One-to-one

Definition
A function $f$ is one-to-one or injective (đơn ánh) if and only if

$$\forall a \forall b \ (f(a) = f(b) \rightarrow a = b)$$
One-to-one

Definition
A function $f$ is one-to-one or injective (đơn ánh) if and only if

$$\forall a \forall b \ (f(a) = f(b) \implies a = b)$$

- Is $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x + 1$ one-to-one?
- Is $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2$ one-to-one?
Onto

**Definition**

$f : A \rightarrow B$ is **onto** or **surjective** (*toàn ánh*) if and only if

\[ \forall b \in B, \exists a \in A : f(a) = b \]

- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x + 1$ onto?
- Is $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^2$ onto?
One-to-one and onto (bijection)

**Definition**

\( f : A \rightarrow B \) is **bijective** (one-to-one correspondence) \((song \ ánh)\) if and only if \( f \) is **injective** and **surjective**

Let \( f \) be the function from \( \{a, bc, d\} \) to \( \{1, 2, 3, 4\} \) with \( f(a) = 4, f(b) = 2, f(c) = 1, f(d) = 3 \). Is \( f \) a bijection?
Example

4.12

Functions

Tran Vinh Tan

Contents

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One-to-one and Onto Functions

Sequences and Summation

Recursion

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Example
Example
Example

\begin{itemize}
\item \textbf{One-to-one and Onto Functions}
\item \textbf{Sequences and Summation}
\item \textbf{Recursion}
\end{itemize}

[Diagrams showing examples of one-to-one, onto, and other function types]
Example

- One-to-one and Onto Functions
- Sequences and Summation
- Recursion
Inverse function (Hàm ngược)

**Definition**

Let \( f : A \to B \) be a **bijection** then the inverse of \( f \) is the function \( f^{-1} : B \to A \) defined by

\[
\text{if } f(a) = b \text{ then } f^{-1}(b) = a
\]

A one-to-one correspondence is call invertible (khả nghịch) because we can define the inverse of this function.
Example

$A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ with

- $f(a) = 2$
- $f(b) = 3$
- $f(c) = 1$

$f$ is invertible and its inverse is

- $f^{-1}(1) = c$
- $f^{-1}(2) = a$
- $f^{-1}(3) = b$
Example

$A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ with

$$f(a) = 2 \quad f(b) = 3 \quad f(c) = 1$$

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Example

Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$. Is $f$ invertible?
Example

\[ f : \mathbb{R} \rightarrow \mathbb{R} \]
\[ f(x) = 2x + 1 \]

\[ f^{-1} : \mathbb{R} \rightarrow \mathbb{R} \]
\[ f^{-1}(x) = x - 1 \]
Example

\[ f : \mathbb{R} \to \mathbb{R} \]

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\( f : \mathbb{R} \to \mathbb{R} \)

\[ f(x) = 2x + 1 \]

\( f^{-1} : \mathbb{R} \to \mathbb{R} \)

\[ f^{-1}(x) = \frac{x - 1}{2} \]
Function Composition

**Definition**

Given a pair of functions $g : A \rightarrow B$ and $f : B \rightarrow C$. Then the composition (hợp thành) of $f$ and $g$, denoted $f \circ g$ is defined by

$$f \circ g : A \rightarrow C$$

$$f \circ g(a) = f(g(a))$$
Example

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Example

\[ X \rightarrow Y \rightarrow Z \]

1 \rightarrow D \rightarrow P
2 \rightarrow B \rightarrow Q
3 \rightarrow C \rightarrow R
A \rightarrow S
Example

\[ X \rightarrow Y \rightarrow Z \]

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\[ 3 \rightarrow C \rightarrow R \]

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Example


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Graphs of Functions

Example

The graph of \( f(x) = x^2 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \).
Example

The graph of $f(x) = x^2$ from $\mathbb{Z}$ to $\mathbb{Z}$.

\[
\begin{align*}
(-3, 9) & \quad (3, 9) \\
(-2, 4) & \quad (2, 4) \\
(-1, 1) & \quad (1, 1) \\
(0, 0) & \\
\end{align*}
\]
**Example**

The graph of \( f(x) = x^2 \) from \( \mathbb{Z} \) to \( \mathbb{Z} \).

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(0, 0) & 
\end{align*}
\]

**Definition**

Let \( f \) be a function from the set \( A \) to the set \( B \). The **graph** of the function \( f \) is the set of ordered pairs \( \{(a, b) \mid a \in A \text{ and } f(a) = b\} \).
Important Functions

Definition

Floor function \((hàm sàn)\) of \(x\) \((\lfloor x \rfloor)\): the largest integer \(\leq x\)
\[\frac{1}{2} = 0, \quad [3.1] = 3, \quad [7] = 7\]
Important Functions

**Definition**

Floor function (hàm sàn) of \( x \) ([\( x \)]): the largest integer \( \leq x \)

\[
\left\lfloor \frac{1}{2} \right\rfloor = 0, \quad \left\lfloor 3.1 \right\rfloor = 3, \quad \left\lfloor 7 \right\rfloor = 7
\]

Ceiling function (hàm trần) of \( x \) (\( \lceil x \rceil \)): the smallest integer \( \geq x \)

\[
\left\lceil \frac{1}{2} \right\rceil = 1, \quad \left\lceil 3.1 \right\rceil = 4, \quad \left\lceil 7 \right\rceil = 7
\]
Important Functions

Definition

**Floor function** *(hàm sàn)* of *x* ([*x*]): the largest integer ≤ *x*

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\left\lfloor \frac{1}{2} \right\rfloor = 0, \left\lfloor 3.1 \right\rfloor = 3, \left\lfloor 7 \right\rfloor = 7
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**Ceiling function** *(hàm trần)* of *x* ([*x*]): the smallest integer ≥ *x*

\[
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\]

**Bảng: Properties** *(n is an integer, x is a real number)*

<table>
<thead>
<tr>
<th>Condition</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1a) [x] = <em>n</em> iff <em>n</em> ≤ <em>x</em> &lt; <em>n</em> + 1</td>
<td></td>
</tr>
<tr>
<td>(1b) [x] = <em>n</em> iff <em>n</em> − 1 &lt; <em>x</em> ≤ <em>n</em></td>
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<tr>
<td>(1c) [x] = <em>n</em> iff <em>x</em> − 1 &lt; <em>n</em> ≤ <em>x</em></td>
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<td>(1d) [x] = <em>n</em> iff <em>x</em> ≤ <em>n</em> &lt; <em>x</em> + 1</td>
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Important Functions

**Definition**

Floor function \((hàm sàn)\) of \(x\) \(((x)\)): the largest integer \(\leq x\)

\[
\left\lfloor \frac{1}{2} \right\rfloor = 0, \left\lfloor 3.1 \right\rfloor = 3, \left\lfloor 7 \right\rfloor = 7
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Ceiling function \((hàm trần)\) of \(x\) \(((x)\)): the smallest integer \(\geq x\)

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\left\lceil \frac{1}{2} \right\rceil = 1, \left\lceil 3.1 \right\rceil = 4, \left\lceil 7 \right\rceil = 7
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**Bảng:** Properties \((n\) is an integer, \(x\) is a real number\))

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Important Functions

**Definition**

**Floor function** (*hàm sàn*) of $x$ ($\lfloor x \rfloor$): the largest integer $\leq x$

$\lfloor \frac{1}{2} \rfloor = 0$, $\lfloor 3.1 \rfloor = 3$, $\lfloor 7 \rfloor = 7$

**Ceiling function** (*hàm trần*) of $x$ ($\lceil x \rceil$): the smallest integer $\geq x$

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**Bảng:** Properties ($n$ is an integer, $x$ is a real number)

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Sequences

What are the rule of these sequences (dãy)?

Arithmetic sequence (cấp số cộng)

Example

\[1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\]

\[a_n = \frac{1}{2^n} - 1\]

Geometric sequence (cấp số nhân)

Example

\[\{a_n\} = 5, 11, 17, 23, 29, 35, 41, 47, \ldots\]

\[a_n = 6n - 1\]

\[\{b_n\} = 1, 7, 25, 79, 241, 727, 2185, \ldots\]

\[b_n = 3^n - 2\]
Sequences

What are the rule of these sequences (dãy)?

Example

1, 3, 5, 7, 9, ...

\[ a_n = 2^n - 1 \]

Arithmetic sequence (cấp số cộng)

Example

1, 1.5, 2, 2.5, 3, 3.5, ...

\[ a_n = \frac{1}{2}n - 1 \]

Geometric sequence (cấp số nhân)

Example

\{a_n\} 5, 11, 17, 23, 29, 35, 41, ...

\[ a_n = 6^n - 1 \]

\{b_n\} 1, 7, 25, 79, 241, 727, 2185, ...

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Sequences

What are the rule of these sequences (dãy)?

Example

1, 3, 5, 7, 9, ... \( a_n = 2n - 1 \)

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1, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \)
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**Example**

1, 3, 5, 7, 9, ... \( a_n = 2n - 1 \)

**Example**

1, \( \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots \) \( a_n = \frac{1}{2^{n-1}} \)
Sequences

What are the rule of these sequences (dãy)?

**Example**

1, 3, 5, 7, 9, ...  \( a_n = 2n - 1 \)

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**Example**

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What are the rule of these sequences (dãy)?

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\(\{a_n\}\)  5, 11, 17, 23, 29, 35, 41, 47, ...  \(a_n = 6n - 1\)

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<table>
<thead>
<tr>
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Find the Fibonacci numbers \(f_2, f_3, f_4, f_5\) and \(f_6\)

\[f_2 = f_1 + f_0 = 1 + 0 = 1\]
\[f_3 = f_2 + f_1 = 1 + 1 = 2\]
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Initial deposit: $10,000
Interest: 11%/year, compounded annually (lãi suất kép)

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Let $P_n$ be the amount in the account after $n$ years. The sequence \(\{P_n\}\) satisfies the recurrence relation
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\[ P_1 = (1.11)P_0 \]
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Step 2. Calculate
\[ P_{30} = (1.11)^{30}10,000 = $228,922.97. \]
Exercise (2)

What is the 2012th number in the sequence \( \{x_n\} \): 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, \ldots 

Solution:
In this sequence, integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and so on. Therefore integer \( n \) appears \( n \) times in the sequence.

We can prove that \( \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \) and can easily calculate that \( \sum_{i=1}^{62} i = 1953 \) so the next 63 numbers (until 2016) is 63.

Therefore, 2012th number in the sequence is 63.
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Theorem

If $a$ and $r$ are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$
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$$

증명.

Let $S_n = \sum_{j=0}^{n} ar^j$.

$$
rS_n = r \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} ar^{j+1} = \sum_{k=1}^{n+1} ar^{k} = \left( \sum_{k=0}^{n} ar^{k} \right) + (ar^{n+1} - a) = S_n + (ar^{n+1} - a)
$$

Solving for $S_n$ shows that if $r \neq 1$, then $S_n = \frac{ar^{n+1}-a}{r-1}$

If $r = 1$, then $S_n = \sum_{j=0}^{n} a = (n+1)a$
### Recursion

**Definition (Recurrence Relation)**

An equation that *recursively defines* a sequence.
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An algorithm is called **recursive** if it solves a problem by reducing it to an instance of the same problem with smaller input.
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**Example**
Give a recursive algorithm for computing $n!$, where $n$ is a nonnegative integer.

procedure factorial (n: nonnegative integer)
if $n = 0$
then return 1
else return $n \cdot$ factorial ($n - 1$)
{output is $n!$}
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$n! = n \cdot (n - 1)!$ and $0! = 1$. 
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procedure factorial($n$: nonnegative integer)
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Algorithms for Fibonacci Numbers

## Contents

- Functions
  - One-to-one and Onto Functions
  - Sequences and Summation
  - Recursion

### 4.27 Algorithms for Fibonacci Numbers

#### Recursive Algorithm

```plaintext
procedure fibonacci(n: nonnegative integer)
if n = 0 then return 0
else if n = 1 then return 1
else return fibonacci(n-1) + fibonacci(n-2)
{output is fibonacci(n)}
```

#### Iterative Algorithm

```plaintext
procedure iterative fibonacci(n: nonnegative integer)
if n = 0 then return 0
else
  x := 0
  y := 1
  for i := 1 to n-1
    z := x + y
    x := y
    y := z
  return y
{output is the nth Fibonacci number}
```

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Algorithms for Fibonacci Numbers

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There is a tower in Hanoi that has three pegs mounted on a board together with 64 gold disks of different sizes. Initially, these disks are placed on the first peg in order of size, with the largest on the bottom.
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The myth says that the world will end when they finish the puzzle.
Tower of Hanoi – 64 Discs
Tower of Hanoi – 1 Disc
Tower of Hanoi – 1 Disc

Moved disc from peg 1 to peg 3.
Tower of Hanoi – 1 Disc
Tower of Hanoi – 2 Discs
Tower of Hanoi – 2 Discs

Moved disc from peg 1 to peg 2.
Tower of Hanoi – 2 Discs

Moved disc from peg 1 to peg 3.
TOWER OF HANOI – 2 DISCS

Moved disc from peg 2 to peg 3.
Tower of Hanoi – 2 Discs
Tower of Hanoi – 3 Discs
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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs

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Tower of Hanoi – 3 Discs
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Tower of Hanoi – 4 Discs

Moved disc from peg 2 to peg 3.
Tower of Hanoi – 4 Discs
Tower of Hanoi

Algorithm

```python
procedure hanoi(n, A, B, C)
if \( n = 1 \) then move the disk from A to C
else
  call hanoi(n − 1, A, C, B)
  move disk \( n \) from A to C
  call hanoi(n − 1, B, A, C)
```

Recurrence Relation

\[
H(n) = \begin{cases} 
  1 & \text{if } n = 1 \\
  2H(n − 1) + 1 & \text{if } n > 1.
\end{cases}
\]

Recurrence Solving

\[
H(n) = 2^n − 1
\]

If one move takes 1 second, for \( n = 64 \)

\[
2^{64} − 1 \approx 2 \times 10^{19} \text{ sec}
\approx 500 \text{ billion years}!
\]