Chapter 8 - Heaps

- Efficient implementation of heap ADT: use of array
- Basic heap algorithms: ReheapUp, ReheapDown, Insert Heap, Delete Heap, Built Heap
- d-heaps
- Heap Applications:
  - Select Algorithm
  - Priority Queues
  - Heap sort
- Advanced implementations of heaps: use of pointers
  - Leftist heap
  - Skew heap
  - Binomial queues
**DEFINITION**: A **max-heap** is a binary tree structure with the following properties:
- The tree is complete or nearly complete.
- The key value of each node is greater than or equal to the key value in each of its descendants.

**DEFINITION**: A **min-heap** is a binary tree structure with the following properties:
- The tree is complete or nearly complete.
- The key value of each node is less than or equal to the key value in each of its descendents.
Properties of Binary Heaps

- Structure property of heaps
- Key value order of heaps

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Properties of Binary Heaps

Structure property of heaps:

• A complete or nearly complete binary tree.

• If the height is $h$, the number of nodes $n$ is between $2^{h-1}$ and $(2^h - 1)$

• Complete tree: $n = 2^h - 1$ when last level is full.

• Nearly complete: All nodes in the last level are on the left.

$h = \lfloor \log_2 n \rfloor + 1$

• Can be represented in an array and no pointers are necessary.
Properties of Binary Heaps

Key value order of max-heap:

(max-heap is often called as heap)
Basic heap algorithms

ReheapUp: repairs a "broken" heap by floating the last element up the tree until it is in its correct location.
Basic heap algorithms

**ReheapDown**: repairs a "broken" heap by pushing the root of the subtree down until it is in its correct location.
Contiguous Implementation of Heaps

Heap

data <Array of <DataType> >
count <int> //number of elements in heap

End Heap

Conceptual

Physical

[(i-1)/2]
Algorithm **ReheapUp** (val position <int>)

Reestablishes heap by moving data in position up to its correct location.

**Pre**  All data in the heap above this position satisfy key value order of a heap, except the data in position.

**Post** Data in position has been moved up to its correct location.

**Uses** Recursive function ReheapUp.

1. if (position <> 0) // the parent of position exists.
   1. parent = (position-1)/2
   2. if (data[position].key > data[parent].key)
      1. swap(position, parent) // swap data at position with data at parent.
      2. ReheapUp(parent)
   2. return

End ReheapUp
Algorithm **ReheapDown** (val position <int>, val lastPosition <int>)

Reestablishes heap by moving data in position down to its correct location.

**Pre** All data in the subtree of position satisfy key value order of a heap, except the data in position.

**Post** Data in position has been moved down to its correct location.

**Uses** Recursive function **ReheapDown**.

1. leftChild = position * 2 + 1
2. rightChild = position * 2 + 2
3. if (leftChild <= lastPosition) // the left child of position exists.
   1. if (rightChild <= lastPosition) AND (data[rightChild].key > data[leftChild].key)
      1. child = rightChild
   2. else
      1. child = leftChild // choose larger child to compare with data in position
3. if (data[child].key > data[position].key)
   1. swap(child, position) // swap data at position with data at child.
   2. ReheapDown(child, lastPosition)
4. return

End ReheapDown
Insert new element into min-heap

Insert 14:

The new element is put to the last position, and ReheapUp is called for that position.
**InsertHeap** (val DataIn <DataType>) // Recursive version.

Inserts new data into the min-heap.

**Post**  
DataIn has been inserted into the heap and the heap order property is maintained.

**Return**  
*overflow* or *success*

**Uses**  
recursive function ReheapUp.

1. if (heap is full)
   1. return *overflow*
2. else
   1. data[count] = DataIn
   2. ReheapUp(count)
   3. count = count + 1
   4. return *success*

End InsertHeap
**InsertHeap** (val DataIn <DataType>)  // Iterative version

Inserts new data into the min-heap.

**Post**  
DataIn has been inserted into the heap and the heap order property is maintained.

**Return**  
*overflow* or *success*

1. if (heap is full)
   1. return *overflow*

2. else
   1. current_position = count - 1
   2. loop (the parent of the element at the current_position is exists) AND (parent.key > DataIn.key)
      1. data[current_position] = parent
      2. current_position = position of parent
   3. data[current_position] = DataIn
   4. count = count + 1
   5. return *success*

End InsertHeap
Delete minimum element from min-heap

The element in the last position is put to the position of the root, and 
ReheapDown is called for that position.
Delete minimum element from min-heap

The element in the last position is put to the position of the root, and ReheapDown is called for that position.
DeleteHeap (ref MinData <DataType>)  // Recursive version

Removes the minimum element from the min-heap.

**Post** MinData receives the minimum data in the heap and this data has been removed. The heap has been rearranged.

**Return** underflow or success

**Uses** recursive function ReheapDown.

1. if (heap is empty)
   1. return underflow
2. else
   1. MinData = Data[0]
   2. Data[0] = Data[count - 1]
   3. count = count - 1
   4. ReheapDown(0, count - 1)
   5. return success

End DeleteHeap
DeleteHeap (ref MinData <DataType>) // Iterative version

Removes the minimum element from the min-heap.

Post MinData receives the minimum data in the heap and this data has been removed. The heap has been rearranged.

Return underflow or success

1. if (heap is empty)
   1. return underflow
2. else
   1. MinData = Data[0]
   2. lastElement = Data[count – 1] // The number of elements in the heap is decreased so the last element must be moved somewhere in the heap.
// DeleteHeap (cont.) // Iterative version

3. current_position = 0
4. continue = TRUE
5. loop (the element at the current_position has children) AND (continue = TRUE)

1. Let child is the smaller of two children
2. if (lastElement.key > child.key )
   1. Data[current_position] = child
   2. current_position = current_position of child
3. else
   1. continue = FALSE

6. Data[current_position] = lastElement
7. count = count - 1
8. return success

End DeleteHeap
Build heap

<ErrorCode> BuildHeap (val listOfData <List>)
Builds a heap from data from listOfData.

Pre    listOfData contains data need to be inserted into an empty heap.
Post   Heap has been built.
Return overflow or success
Uses   Recursive function ReheapUp.

1. count = 0
2. loop (heap is not full) AND (more data in listOfData)
   1. listOfData.Retrieve(count, newData)
   2. data[count] = newData
   3. ReheapUp( count)
   4. count = count + 1
3. if (count < listOfData.Size() )
   1. return overflow
4. else
   1. return success
End BuildHeap
Algorithm **BuildHeap2** ()

Builds a heap from an array of random data.

**Pre**  Array of `count` random data.

**Post**  Array of data becomes a heap.

**Uses**  Recursive function **ReheapDown**.

1. `position = count / 2 - 1`
2. **loop** (`position >= 0`)
   1. **ReheapDown** (`position, count-1`)
   2. `position = position - 1`
3. `return`

End **BuildHeap2**
Complexity of Binary Heap Operations

- ReheapUp: $O(\log_2 n)$
- ReheapDown: $O(\log_2 n)$
- BuildHeap: $O(n \log_2 n)$
- InsertHeap: $O(\log_2 n)$
- DeleteHeap: $O(\log_2 n)$
d-heaps

- d-heap is a simple generalization of a binary heap.
- In d-heap, all nodes have $d$ children.
- d-heap improve the running time of InsertElement to $O(\log_d n)$.
- For large $d$, DeleteMin operation is more expensive: the minimum of $d$ children must be found, which takes $d-1$ comparisons.
- The multiplications and divisions to find children and parents are now by $d$, which increases the running time. (If $d=2$, use of the bit shift is faster).
- d-heap is suitable for the applications where the number of Insertion is greater than the number of DeleteMin.
Heap Applications

- Select Algorithms.
- Priority Queues.
- Heap sort (*we will see in the Sorting Chapter*).
Determine the $k^{th}$ largest element in an unsorted list

**Algorithm 1a:**

- Read the elements into an array, sort them.
- Return the appropriate element.

*The running time of a simple sorting algorithm is $O(n^2)$*
Select Algorithms

Determine the \( k^{th} \) largest element in an unsorted list

**Algorithm 1b:**

- Read \( k \) elements into an array, sort them.
- The smallest of these is in the \( k^{th} \) position.
- Process the remaining elements one by one.
- Compare the coming element with the \( k^{th} \) element in the array.
- If the coming element is large, the \( k^{th} \) element is removed, the new element is placed in the correct place.

*The running time is \( O(n^2) \)*
Select Algorithms

Determine the $k^{th}$ largest element in an unsorted list

Algorithm 2a:

- Build a max-heap.
- Delete $k-1$ elements from the heap.
- The desired element will be at the top.

The running time is $O(n\log_2 n)$
Select Algorithms

Determine the $k^{th}$ largest element in an unsorted list

**Algorithm 2b:**

- Build a min-heap of $k$ elements.
- Process the remaining elements one by one.
- Compare the coming element with the minimum element in the heap (the element on the root of heap).
- If the coming element is large, the minimum element is removed, the new element is placed in the correct place (reheapdown).

*The running time is $O(n\log_2 n)$*
Priority Queue ADT

- Jobs are generally placed on a queue to wait for the services.
- In the multiuser environment, the operating system scheduler must decide which of several processes to run.
- Short jobs finish as fast as possible, so they should have precedence over other jobs.
- Otherwise, some jobs are still very important and should also have precedence.

These applications require a special kind of queue: a priority queue.
Priority Queue ADT

- Each element has a priority to be dequeued.
- Minimum value of key has highest priority order.

**DEFINITION of Priority Queue ADT:**
Elements are enqueued accordingly to their priorities.
Minimum element is dequeued first.

**Basic Operations:**
- *Create*
- *InsertElement*: Inserts new data to the position accordingly to its priority order in queue.
- *DeleteMin*: Removes the data with highest priority order.
- *RetrieveMin*: Retrieves the data with highest priority order.
Priority Queue ADT

Extended Operations:

- **Clear**
- **isEmpty**
- **isFull**
- **RetrieveMax**: Retrieves the data with lowest priority order.
- **IncreasePriority**: Changes the priority of some data which has been inserted in queue.
- **DecreasePriority**
- **DeleteElement**: Removes some data out of the queue.
Specifications for Priority Queue ADT

<ErrorCode> InsertElement (val DataIn <DataType>)
<ErrorCode> DeleteMin (ref MinData <DataType>)
<ErrorCode> RetrieveMin (ref MinData <DataType>)
<ErrorCode> RetrieveMax (ref MaxData <DataType>)
<ErrorCode> IncreasePriority (val position <int>,
                            val PriorityDelta <KeyType>)
<ErrorCode> DecreasePriority (val position <int>,
                             val PriorityDelta <KeyType>)
<ErrorCode> DeleteElement (val position <int>,
                           ref DataOut <DataType>)

<bool> isEmpty()
<bool> isFull()
<void> clear()
Implementations of Priority Queue

- **Use linked list:**

  - **Simple linked list:**
    - Insertion performs at the front, requires $O(1)$.
    - DeleteMin requires $O(n)$ for searching of the minimum data.
  
  - **Sorted linked list:**
    - Insertion requires $O(n)$ for searching of the appropriate position.
    - DeleteMin requires $O(1)$. 
Implementations of Priority Queue

- **Use BST:**
  - Insertion requires $O(\log_2 n)$.  
  - DeleteMin requires $O(\log_2 n)$.  
  - But DeleteMin, repeatedly removing node in the left subtree, seem to hurt **balance of the tree.**
Implementations of Priority Queue

➤ Use min-heap:

• Insertion requires $O(\log_2 n)$.
• DeleteMin requires $O(\log_2 n)$. 
Insert and Remove element into/from priority queue

<ErrorCode> **InsertElement** (val DataIn <DataType>):

InsertHeap Algorithm

<ErrorCode> **DeleteMin** (ref MinData <DataType>):

DeleteHeap Algorithm
Retrieve minimum element in priority queue

<ErrorCode> RetrieveMin (ref MinData <DataType>)
Retrieves the minimum element in the heap.

Post  MinData receives the minimum data in the heap and the heap remains unchanged.
Return  underflow or success

1.   if (heap is empty)
   1.   return underflow
2.   else
   1.   MinData = Data[0]
   2.   return success
End RetrieveMin
Retrieve maximum element in priority queue

<ErrorCode> RetrieveMax (ref MaxData <DataType>)
Retrieves the maximum element in the heap.

Post MaxData receives the maximum data in the heap and the heap remains unchanged.

Return underflow or success

1. if (heap is empty)
   1. return underflow

2. else
   1. Sequential search the maximum data in the right half elements of the heap (the leaves of the heap). The first leaf is at the position count/2.
   2. return success

End RetrieveMax
<ErrorCode> IncreasePriority (val position <int>,
     val PriorityDelta <KeyType>)

Increases priority of an element in the heap.

Post Element at position has its priority increased by PriorityDelta
     and has been moved to correct position.

Return rangeError or success

Uses ReheapDown.
DecreasePriority (val position <int>,
val PriorityDelta <KeyType>)

Decreases priority of an element in the heap.

Post Element at position has its priority decreased by PriorityDelta and has been moved to correct position.

Return rangeError or success

Uses ReheapUp.
Remove an element out of priority queue

`<ErrorCode> DeleteElement (val position <int>,
ref DataOut <DataType>)`

Removes an element out of the min-heap.

Post  DataOut contains data in the element at `position`, this element has been removed. The heap has been rearranged.

Return  `rangeError` or `success`

1.  if (position>=`count`) OR (position <0)
   1.  return `rangeError`
2.  else
   1.  DataOut = Data[position]
   2.  DecreasePriority(position, VERY_LARGE_VALUE),
   3.  DeleteMin(MinData)
   4.  return `success`

End DeleteElement
Advanced implementations of heaps

- Advanced implementations of heaps: use of pointers
  - Leftist heap
  - Skew heap
  - Binomial queues

Use of pointers allows the merge operations (combine two heaps into one) to perform easily.