Chapter 2
Logics (cont.)

Discrete Mathematics I on 08 March 2011

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2 Proof Methods
Limits of Propositional Logic

- $x > 3$
- All square numbers are not prime numbers. 100 is a square number. Therefore 100 is not a prime number.
**Predicates**

**Definition**
A predicate (vị từ) is a statement containing one or more variables. If values are assigned to all the variables in a predicate, the resulting statement is a proposition (mệnh đề).

Example:
- $x > 3$ (predicate)
- $5 > 3$ (proposition)
- $2 > 3$ (proposition)
Predicates

- \( x > 3 \rightarrow P(x) \)
- \( 5 > 3 \rightarrow P(5) \)
- A predicate with \( n \) variables \( P(x_1, x_2, \ldots, x_n) \)
Truth value

- $x > 3$ is true or false?
- $5 > 3$
- For every number $x$, $x > 3$ holds
- There is a number $x$ such that $x > 3$
Quantifiers

- **∀**: Universal – Với mọi
  - \( \forall x P(x) = P(x) \) is T for all \( x \)
- **∃**: Existential – Tồn tại
  - \( \exists x P(x) = \) There exists an element \( x \) such that \( P(x) \) is T
- We need a **domain of discourse** for variable

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Predicate Logic
Proof Methods

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Example

Let $P(x)$ be the statement “$x < 2$”. What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real number?

- $P(3) = 3 < 2$ is false
- $\Rightarrow \forall x P(x)$ is false

- 3 is a counterexample (phan ví dụ) of $\forall x P(x)$

Example

What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real number?
Example

Express the statement “Some student in this class comes from Central Vietnam.”

Solution 1

- $M(x) = x$ comes from Central Vietnam
- Domain for $x$ is the students in the class
- $\exists x M(x)$

Solution 2

- Domain for $x$ is all people
- ...
## Negation of Quantifiers

<table>
<thead>
<tr>
<th>Statement</th>
<th>Negation</th>
<th>Equivalent form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x P(x)$</td>
<td>$\neg (\forall x P(x))$</td>
<td>$\exists x \neg P(x)$</td>
</tr>
<tr>
<td>$\exists x P(x)$</td>
<td>$\neg (\exists x P(x))$</td>
<td>$\forall x \neg P(x)$</td>
</tr>
</tbody>
</table>

### Example

- All CSE students study Discrete Math 1
- Let $C(x)$ denote “$x$ is a CSE student”
- Let $S(x)$ denote “$x$ studies Discrete Math 1”
- $\forall x : C(x) \rightarrow S(x)$
- $\exists x : \neg(C(x) \rightarrow S(x)) \equiv \exists x : C(x) \land \neg S(x)$
- There is a CSE student who does not study Discrete Math 1.
Another Example

Example

Translate these:

- All lions are fierce.
- Some lions do not drink coffee.
- Some fierce creatures do not drink coffee.

Solution

Let $P(x)$, $Q(x)$ and $R(x)$ be the statements “$x$ is a lion”, “$x$ is fierce” and “$x$ drinks coffee”, respectively.

- $\forall x (P(x) \rightarrow Q(x))$.
- $\exists x (P(x) \land \neg R(x))$.
- $\exists x (Q(x) \land \neg R(x))$. 
The Order of Quantifiers

- The order of quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.
- Read from left to right, apply from inner to outer.

**Example**

\[ \forall x \ \forall y \ (x + y = y + x) \]

T for all \( x, y \in \mathbb{R} \)

**Example**

\[ \forall x \ \exists y \ (x + y = 0) \text{ is T,} \]

while

\[ \exists y \ \forall x \ (x + y = 0) \text{ is F} \]
Translating Nested Quantifiers

Example

\[ \forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y))) \]

Provided that:

- \( C(x) \): \( x \) has a computer,
- \( F(x, y) \): \( x \) and \( y \) are friends,
- \( x, y \in \) all students in your school.

Answer

For every student \( x \) in your school, \( x \) has a computer or there is a student \( y \) such that \( y \) has a computer and \( x \) and \( y \) are friends.
Translating Nested Quantifiers

Example

$$\exists x \forall y \forall z \ ((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y, z))$$

Provided that:

- $F(x, y)$: $x, y$ are friends
- $x, y, z \in$ all students in your school.

Answer

There is a student $x$, so that for every student $y$, every student $z$ not the same as $y$, if $x$ and $y$ are friends, and $x$ and $z$ are friends, then $y$ and $z$ are not friends.
Translating into Logical Expressions

Example

1. “There is a student in the class has visited Hanoi”.
2. “Every students in the class have visited Nha Trang or Vung Tau”.

Answer

Assume:

\[ C(x) : x \text{ has visited Hanoi} \]
\[ D(x) : x \text{ has visited Nha Trang} \]
\[ E(x) : x \text{ has visited Vung Tau} \]

We have:

1. \[ \exists x C(x) \]
2. \[ \forall x (D(x) \lor E(x)) \]
Translating into Logical Expressions

Example
Every people has one best friend.

Solution
Assume:
- $B(x, y) : y$ is the best friend of $x$

We have:
$\forall x \exists y \forall z (B(x, y) \land ((y \neq z) \rightarrow \neg B(x, z)))$
Translating into Logical Expressions

Example

If a person is a woman and a parent, then this person is mother of some one.

Solution

We define:

- $C(x) : x$ is woman
- $D(x) : x$ is a parent
- $E(x, y) : x$ is mother of $y$

We have:

$$\forall x((C(x) \land D(x)) \rightarrow \exists y E(x, y))$$
Inference

Example

• If I have a girlfriend, I will take her to go shopping.
• Whenever I and my girlfriend go shopping and that day is a special day, I will surely buy her some expensive gift.
• If I buy my girlfriend expensive gifts, I will eat noodles for a week.
• Today is March 8.
• March 8 is such a special day.
• Therefore, if I have a girlfriend,...
• I will eat noodles for a week.
# Propositional Rules of Inferences

<table>
<thead>
<tr>
<th>Rule of Inference</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>Modus ponens</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td></td>
</tr>
<tr>
<td>( \therefore q )</td>
<td></td>
</tr>
<tr>
<td>( \neg q )</td>
<td>Modus tollens</td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td></td>
</tr>
<tr>
<td>( \therefore \neg p )</td>
<td></td>
</tr>
<tr>
<td>( p \rightarrow q )</td>
<td>Hypothetical syllogism</td>
</tr>
<tr>
<td>( q \rightarrow r )</td>
<td></td>
</tr>
<tr>
<td>( \therefore p \rightarrow r )</td>
<td>(Tam đoạn luận giả định)</td>
</tr>
<tr>
<td>( p \lor q )</td>
<td>Disjunctive syllogism</td>
</tr>
<tr>
<td>( \neg p )</td>
<td></td>
</tr>
<tr>
<td>( \therefore q )</td>
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## Propositional Rules of Inferences

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| \[
\begin{align*}
    p \\
    \therefore p \lor q
\end{align*}
\]            | Addition        |
| \[
\begin{align*}
    p \land q \\
    \therefore p
\end{align*}
\]            | Simplification  |
| \[
\begin{align*}
    p \\
    q \\
    \therefore p \land q
\end{align*}
\]            | Conjunction     |
| \[
\begin{align*}
    p \lor q \\
    \neg p \lor r \\
    \therefore q \lor r
\end{align*}
\]            | Resolution      |

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Example

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution

- $p$: It is raining today
- $q$: We will not have a barbecue today
- $r$: We will have barbecue tomorrow

$p \rightarrow q$
$q \rightarrow r$
\[ \therefore p \rightarrow r \]

Hypothetical syllogism
Example

- It is not sunny this afternoon ($\neg p$) and it is colder than yesterday ($q$)
- We will go swimming ($r$) only if it is sunny
- If we do not go swimming, then we will take a canoe trip ($s$)
- If we take a canoe trip, then we will be home by sunset ($t$)
- We will be home by sunset ($t$)

1. $\neg p \land q$ Hypothesis
2. $\neg p$ Simplification using (1)
3. $r \rightarrow p$ Hypothesis
4. $\neg r$ Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$ Hypothesis
6. $s$ Modus ponens using (4) and (5)
7. $s \rightarrow t$ Hypothesis
8. $t$ Modus ponens using (6) and (7)
**Definition**

Fallacies (ngụy biện) resemble rules of inference but are based on contingencies rather than tautologies.

**Example**

If you do correctly every questions in mid-term exam, you will get 10 grade. You got 10 grade.

Therefore, you did correctly every questions in mid-term exam.

Is \([p \rightarrow q \land q] \rightarrow p\) a tautology?
Rules of Inference for Quantified Statements

<table>
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<tbody>
<tr>
<td>(\forall x P(x))</td>
<td>Universal instantiation ((Cụ thể hóa phổ quát))</td>
</tr>
<tr>
<td>(\therefore P(c))</td>
<td></td>
</tr>
</tbody>
</table>

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</thead>
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<tr>
<td>(P(c)) for an arbitrary (c)</td>
<td>Universal generalization ((Tổng quát hóa phổ quát))</td>
</tr>
<tr>
<td>(\therefore \forall x P(x))</td>
<td></td>
</tr>
</tbody>
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</tr>
</thead>
<tbody>
<tr>
<td>(\exists x P(x))</td>
<td>Existential instantiation ((Cụ thể hóa tồn tại))</td>
</tr>
<tr>
<td>(\therefore P(c)) for some element (c)</td>
<td></td>
</tr>
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<tr>
<td>(P(c)) for some element (c)</td>
<td>Existential generalization ((Tổng quát hóa tồn tại))</td>
</tr>
<tr>
<td>(\therefore \exists x P(x))</td>
<td></td>
</tr>
</tbody>
</table>
Example

- A student in this class has not gone to class
- Everyone in this class passed the first exam
- Someone who passed the first exam has not gone to class

Hint

- $C(x)$: $x$ is in this class
- $B(x)$: $x$ has gone to class
- $P(x)$: $x$ passed the first exam
- Premises???
1. $\exists x (C(x) \land \neg B(x))$ \hspace{1em} \text{Premise}
2. $C(a) \land \neg B(a)$ \hspace{1em} \text{Existential instantiation from (1)}
3. $C(a)$ \hspace{1em} \text{Simplification from (2)}
4. $\forall x (C(x) \rightarrow P(x))$ \hspace{1em} \text{Premise}
5. $C(a) \rightarrow P(a)$ \hspace{1em} \text{Universal instantiation from (4)}
6. $P(a)$ \hspace{1em} \text{Modus ponens from (3) and (5)}
7. $\neg B(a)$ \hspace{1em} \text{Simplification from (2)}
8. $P(a) \land \neg B(a)$ \hspace{1em} \text{Conjunction from (6) and (7)}
9. $\exists x (P(x) \land \neg B(x))$ \hspace{1em} \text{Existential generalization from (8)}
**Introduction**

**Definition**

A proof is a sequence of logical deductions from
- axioms, and
- previously proved theorems
that concludes with a new theorem.
Terminology

- **Theorem** (định lý) = a statement that can be shown to be true
- **Axiom** (tiên đề) = a statement we assume to be true
- **Hypothesis** (giả thiết) = the premises of the theorem
- **Lemma** (*bổ đề*) = less important theorem that is helpful in the proofs of other results
- **Corollary** (*hệ quả*) = a theorem that can be established directly from a proved theorem
- **Conjecture** (*phỏng đoán*) = statement being proposed to be true, when it is proved, it becomes theorem
Proving a Theorem

Many theorem has the form $\forall x P(x) \rightarrow Q(x)$

Goal:
- Show that $P(c) \rightarrow Q(c)$ is true with arbitrary $c$ of the domain
- Apply universal generalization

$\Rightarrow$ How to show that conditional statement $p \rightarrow q$ is true.
Methods of Proof

- Direct proofs (*chứng minh trực tiếp*)
- Proof by contraposition (*chứng minh phản đảo*)
- Proof by contradiction (*chứng minh phản chứng*)
- Mathematical induction (*quy nạp toán học*)
**Direct Proofs**

**Definition**

A direct proof shows that $p \rightarrow q$ is true by showing that if $p$ is true, then $q$ must also be true.

**Example**

**Ex.:** If $n$ is an odd integer, then $n^2$ is odd.

**Pr.:** Assume that $n$ is odd. By the definition, $n = 2k + 1$, $k \in \mathbb{Z}$.

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

is an odd number.
Proof by Contraposition

**Definition**

\[ p \rightarrow q \] can be proved by showing (directly) that its contrapositive, 
\[ \neg q \rightarrow \neg p, \] is true.

**Example**

**Ex.:** If \( n \) is an integer and \( 3n + 2 \) is odd, then \( n \) is odd.

**Pr.:** Assume that “If \( 3n + 2 \) is odd, then \( n \) is odd” is false; or \( n \) is even, so \( n = 2k, k \in \mathbb{Z} \). Substituting

\[ 3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) \]

is even. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, Q.E.D.
Proofs by Contradiction

**Definition**

$p$ is true if if can show that $\neg p \rightarrow (r \land \neg r)$ is true for some proposition $r$.

**Example**

Ex.: Prove that $\sqrt{2}$ is irrational.

Pr.: Let $p$ is the proposition “$\sqrt{2}$ is irrational”. Suppose $\neg p$ is true, which means $\sqrt{2}$ is rational. If so, $\exists a, b \in \mathbb{Z}, \sqrt{2} = a/b$, $a, b$ have no common factors. Squared, $2 = a^2/b^2$, $2b^2 = a^2$, so $a^2$ is even, and $a$ is even, too. Because of that $a = 2c, c \in \mathbb{Z}$. Thus, $2b^2 = 4c^2$, or $b^2 = 2c^2$, which means $b^2$ is even and so is $b$. That means 2 divides both $a$ and $b$, contradict with the assumption.
Problem

Assume that we have an infinite domino string, we want to know whether every dominoes will fall, if we only know two things:

1. We can push the first domino to fall
2. If a domino falls, the next one will be fall

We can! Mathematical induction.
Mathematical Induction

Definition (Induction)

To prove that $P(n)$ is true for all positive integers $n$, where $P(n)$ is a propositional function, we complete two steps:

- **Basis Step**: Verify that $P(1)$ is true.
- **Inductive Step**: Show that the conditional statement $P(k) \to P(k + 1)$ is true for all positive integers $k$

Logic form:

$$[P(1) \land \forall k P(k) \to P(k + 1))] \to \forall n P(n)$$

What is $P(n)$ in domino string case?
Example on Induction

Example

Show that if $n$ is a positive integer, then

$$1 + 2 + \ldots + n = \frac{n(n + 1)}{2}.$$  

Solution

Let $P(n)$ be the proposition that sum of first $n$ is $\frac{n(n + 1)}{2}$

- **Basis Step:** $P(1)$ is true, because $1 = \frac{1(1+1)}{2}$

- **Inductive Step:**
  Assume that $1 + 2 + \ldots + k = \frac{k(k+1)}{2}$.

Then:

$$1 + 2 + \ldots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

$$= \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{(k + 1)(k + 2)}{2}$$

shows that $P(k + 1)$ is true under the assumption that $P(k)$ is true.
Example on Induction

Example

Prove that \( n < 2^n \) for all positive integers \( n \).

Solution

Let \( P(n) \) be the proposition that \( n > 2^n \).

- **Basis Step:** \( P(1) \) is true, because \( 1 > 2^1 = 2 \)

- **Inductive Step:**
  Assume that \( P(k) \) is true for the positive \( k \), that is, \( k < 2^k \).
  Add 1 to both side of \( k < 2^k \), note that \( 1 \leq 2^k \).

\[
  k + 1 < 2^k + 1 \leq 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}.
\]

shows that \( P(k + 1) \) is true, namely, that \( k + 1 < 2^{k+1} \),
based on the assumption that \( P(k) \) is true.