DERIVATIVES

Electronic version of lecture

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OUTLINE

1. Derivatives

2. Higher derivatives

3. Linear approximations and differentials

4. Rates of change and related rates

5. Matlab
**Definition 1.1**

The *tangent line* to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through $P$ with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.
EXAMPLE 1.1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION The slope of tangent line to the parabola $y = x^2$ is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

The equation of the tangent line at $(1, 1)$ is

$$y - 1 = 2(x - 1) \iff y = 2x - 1$$
Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where $s$ is the directed distance of the object from the origin at the time $t$. In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$. The **average velocity** over this time interval is

$$\text{average velocity} = \frac{f(a + h) - f(a)}{h}$$
Now suppose we compute the average velocities over shorter and shorter time intervals \([a, a + h]\). We let \(h\) approach 0. The **instantaneous velocity** \(v(a)\) at time \(t = a\) is defined by

\[
v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

(2)
**Example 1.2**

Suppose that a ball is dropped from the upper observation deck of CN Tower, 450 m above the ground.

1. What is the velocity of the ball after 5 seconds?
2. How fast is the ball travelling when it hits the ground?
The equation of motion \( s = f(t) = \frac{1}{2}g.t^2 = 4.9t^2 \), where \( g \)– acceleration of gravity

\[
v(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a + h)^2 - 4.9a^2}{h} = \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a
\]

The velocity after 5s is \( v(5) = 9.8 \times 5 = 49 \text{ m/s} \)

Since the observation deck is 450 m above the ground, the ball will hit the ground at the time \( t_1 \) when \( s(t_1) = 450 \Rightarrow 4.9.t_1^2 = 450 \Rightarrow t_1 = \sqrt{\frac{450}{4.9}} \approx 9.6\text{s} \Rightarrow v(t_1) = 9.8t_1 \approx 94\text{ m/s}. \) (the velocity of the ball as it hits the ground)
**Definition 1.2**

The *derivative of a function* $f$ *at a number* $a$, denoted by $f'(a)$, (read: $f$ prime of $a$) is

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

if this limit exists.

**Other Notations:** $f'(a) = y'(a) = \frac{dy}{dx} \bigg|_{x=a} = \frac{d}{dx} f(a)$. 
**Example 1.3**

*Find the derivative of the function* $f(x) = x^2 - 8x + 9$ *at the number* $a$

**Solution**

\[
\begin{align*}
  f'(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\
  &= \lim_{h \to 0} \frac{[(a + h)^2 - 8(a + h) + 9] - (a^2 - 8a + 9)}{h} \\
  &= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} \\
  &= \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h} \\
  &= \lim_{h \to 0} (2a + h - 8) = 2a - 8
\end{align*}
\]
**Definition 1.3**

The *left-hand derivative* of \( y = f(x) \) at a number \( a \) is the limit (if this limit exists)

\[
f'_-(a) = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \tag{4}
\]

The *right-hand derivative* of \( y = f(x) \) at a number \( a \) is the limit (if this limit exists)

\[
f'_+(a) = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \tag{5}
\]
**Theorem 1.1**

A function $y = f(x)$ is **differentiable at a** if and only if the left-hand and the right-hand derivatives of $f$ at $a$ exist and are equal.

$$f'(a) = f'_-(a) = f'_+(a)$$  \(6\)

**Definition 1.4**

A function $y = f(x)$ is **differentiable on an open interval** $(a, b)$ [or $(a, \infty)$ or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.
**Example 1.4**

Where is the function \( y = f(x) = |x| = \begin{cases} 
  x, & x \geq 0 \\
  -x, & x < 0 
\end{cases} \) not differentiable?

**Solution**

\[
\begin{align*}
  f'_+(0) &= \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^+} \frac{x}{x} = 1 \\
  f'_-(0) &= \lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1
\end{align*}
\]

Since \( f'_+(0) = 1 \neq -1 = f'_-(0) \), \( f'(0) \) does not exist. Thus \( f \) is not differentiable at \( a = 0 \).
1. When $a > 0$ we have

$$f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1$$

2. When $a < 0$ we have

$$f'(a) = \lim_{x \to a} \frac{|x| - |a|}{x - a} = \lim_{x \to a} \frac{-x - (-a)}{x - a} = -1$$

**Conclusion:** $f$ is not differentiable at $a = 0$. 
**Differentiation Formulas I**

1. **Derivative of a constant function**
   
   \[ y = C = \text{const} \Rightarrow y' = 0. \]

2. **Derivatives of power functions**
   
   \[ y = x^\alpha (x \neq 0) \Rightarrow y' = \alpha x^{\alpha-1}. \]

   **Special cases:**
   
   a) \( y = x \Rightarrow y' = 1. \)
   
   b) \( y = \frac{1}{x} \Rightarrow y' = -\frac{1}{x^2}. \)
   
   c) \( y = \sqrt{x} \Rightarrow y' = \frac{1}{2\sqrt{x}}. \)
   
   d) \( y = \sqrt[n]{x} \Rightarrow y' = \frac{1}{n\sqrt[n]{x^{n-1}}}. \)
Derivatives of exponential functions

\[ y = a^x (a > 0, a \neq 1) \Rightarrow y' = a^x \ln a. \]

Special case: \( y = e^x \Rightarrow y' = e^x \), since \( \ln e = 1 \)

Derivatives of logarithmic functions

\[ y = \log_a |x| \ (a > 0, a \neq 1) \Rightarrow y' = \frac{1}{x \ln a}. \]

Special case: \( y = \ln |x| \Rightarrow y' = \frac{1}{x} \) since \( \ln e = 1 \)
Derivatives of trigonometric functions

5. \( y = \sin x \Rightarrow y' = \cos x. \)

6. \( y = \cos x \Rightarrow y' = -\sin x. \)

7. \( y = \tan x \Rightarrow y' = \frac{1}{\cos^2 x} \)

8. \( y = \cot x \Rightarrow y' = -\frac{1}{\sin^2 x} \)
Derivatives of inverse trigonometric functions

9. \( y = \arcsin x, \ (x \in (-1, 1)) \Rightarrow y' = \frac{1}{\sqrt{1 - x^2}} \)

10. \( y = \arccos x, \ (x \in (-1, 1)) \Rightarrow y' = -\frac{1}{\sqrt{1 - x^2}} \)

11. \( y = \arctan x \Rightarrow y' = \frac{1}{1 + x^2} \)

12. \( y = \arccot x \Rightarrow y' = -\frac{1}{1 + x^2} \)
Derivatives of hyperbolic functions

13. \( y = \sinh x \Rightarrow y' = \cosh x \)

14. \( y = \cosh x \Rightarrow y' = \sinh x \)

15. \( y = \tanh x \Rightarrow y' = \frac{1}{\cosh^2 x} \)

16. \( y = \coth x \Rightarrow y' = -\frac{1}{\sinh^2 x} \)
1. The constant multiple rule
   \[ y = cu = cu(x) \Rightarrow y' = cu'(x), \ c - \text{const.} \]

2. The sum (difference) rule
   \[ y = u(x) \pm v(x) \Rightarrow y' = u'(x) \pm v'(x). \]

3. The product rule \[ y = u(x).v(x) \]
   \[ \Rightarrow y' = u'(x).v(x) + u(x).v'(x) \]

4. The quotient rule \[ y = \frac{u(x)}{v(x)} \]
   \[ \Rightarrow y' = \frac{u'(x).v(x) - u(x).v'(x)}{v^2(x)} \]
THE CHAIN RULE

THEOREM 1.2

If function $u = u(x)$ is differentiable at $x$ and function $y = f(u)$ is differentiable at $u(x)$ then the composite function $y = f \circ u = f(u) = f(u(x))$ is differentiable at $x$ and $y'$ is given by the product

$$y'(x) = f'(u(x)).u'(x). \tag{7}$$
**Example 1.5**

Differentiate $y = \sin^5(4x + 3)$

**SOLUTION**

$$y' = 5\sin^4(4x + 3).[\sin(4x + 3)]' =$$

$$= 5\sin^4(4x + 3).\cos(4x + 3).(4x + 3)' =$$

$$= 20\sin^4(4x + 3)\cos(4x + 3).$$
**Definition 2.1**

If $f$ is a differentiable function, then its derivative $f'$ is also a function. If $f'(x)$ has derivative on the interval $(a,b)$ then the derivative of $f'(x)$ is called the second derivative of $f(x)$. It is denoted by $f''(x) = (f'(x))'$

**Example 2.1**

If $f(x) = \frac{2x + 3}{x - 2}$, then find $f''(0)$.

**Solution**

$$f'(x) = \frac{-7}{(x-2)^2} \Rightarrow f''(x) = \frac{14}{(x-2)^3} \Rightarrow f''(0) = -\frac{7}{4}$$
**Example 2.2**

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t)$$

The instantaneous rate of change of velocity with respect to time is called the *acceleration* $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$
**Definition 2.2**

The $n$–th derivative of $f(x)$ is obtained from $f$ by differentiating $n$ times.

$$f^{(n)}(x) = (f^{(n-1)}(x))', \quad n \in \mathbb{N}.$$ (8)

**Properties**

If $f(x)$ and $g(x)$ have $n$–th derivatives then $c_1 f(x) + c_2 g(x), \quad c_1, c_2 \in \mathbb{R}$ also has $n$–th derivative and

$$(c_1 f(x) + c_2 g(x))^{(n)} = c_1 f^{(n)}(x) + c_2 g^{(n)}(x)$$ (9)
Leibniz’s Formula.

If \( f(x) \) and \( g(x) \) have \( n \)–th derivatives then \( f(x).g(x) \) also has \( n \)–th derivative and

\[
(f(x).g(x))^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)}(x)g^{(k)}(x).
\]

(10)
**SOME BASIC FORMULAS**

1. \((a^x)^{(n)} = a^x \cdot \ln^n a.\)
2. \((e^x)^{(n)} = e^x\)
3. \((\sin ax)^{(n)} = a^n \sin \left(ax + \frac{n\pi}{2}\right)\)
4. \((\cos ax)^{(n)} = a^n \cos \left(ax + \frac{n\pi}{2}\right)\)
5. \(((ax + b)^\alpha)^{(n)} = a^n \alpha \alpha (\alpha - 1) \ldots (\alpha - n + 1) (ax + b)^{\alpha - n}\)
6. \((\log_a |x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n \ln a}\)
7. \((\ln |x|)^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}\)
**Example 2.3**

Find the $n$–th derivative of $f(x) = \frac{1}{x^2 - 4}$

**SOLUTION**

\[
\frac{1}{x^2 - 4} = \frac{1}{4} \left( \frac{1}{x - 2} - \frac{1}{x + 2} \right)
\]

It implies

\[
\left( \frac{1}{x^2 - 4} \right)^{(n)} = \frac{1}{4} \left( \left( \frac{1}{x - 2} \right)^{(n)} - \left( \frac{1}{x + 2} \right)^{(n)} \right)
\]
By substituting $\alpha = -1, a = 1, b = \pm 2$, we have

$$
\left( \frac{1}{x \pm 2} \right)^{(n)} = (-1)(-2)\ldots(-1 - n + 1)(x \pm 2)^{-1-n}
$$

Therefore

$$
f^{(n)}(x) = \frac{(-1)^n n!}{4} \left( \frac{1}{(x - 2)^{n+1}} - \frac{1}{(x + 2)^{n+1}} \right)
$$
**Example 2.4**

*Find the $n$–th derivative of $f(x) = x^2 \cos 2x$.

Using Leibniz’s formula, we have

$$(x^2 \cdot \cos 2x)^{(n)} = C_0^n x^2 (\cos 2x)^{(n)} +$$

$$+ C_1^n (x^2)' (\cos 2x)^{(n-1)} + C_2^n (x^2)'' (\cos 2x)^{(n-2)}$$

On another hand, we have

$$(\cos 2x)^{(n)} = 2^n \cos \left(2x + \frac{n\pi}{2}\right),$$
Higher derivatives

Some basic formulas

$$(\cos 2x)^{(n-1)} = 2^{n-1} \cos \left(2x + \frac{(n-1)\pi}{2}\right) =$$

$$= 2^{n-1} \sin \left(2x + \frac{n\pi}{2}\right),$$

$$(\cos 2x)^{(n-2)} = 2^{n-2} \cos \left(2x + \frac{(n-2)\pi}{2}\right) =$$

$$= -2^{n-2} \cos \left(2x + \frac{n\pi}{2}\right).$$

So

$$(x^2 \cdot \cos 2x)^{(n)} = 2^n \left(x^2 - \frac{n(n-1)}{4}\right) \cos \left(2x + \frac{n\pi}{2}\right)$$

$$+ 2^n nx \sin \left(2x + \frac{n\pi}{2}\right).$$
Linear approximations and Differentials

**LINEAR APPROXIMATIONS**

We use the tangent line at \((a, f(a))\) as an approximation to the curve \(y = f(x)\) when \(x\) is near \(a\).
**Definition 3.1**

1. $f(x) \approx f(a) + f'(a)(x - a)$ is called the **linear approximation or tangent line approximation** of $f$ at $a$.

2. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of $f$ at $a$. 
EXAMPLE 3.1

Find the linearization of the function \( f(x) = \sqrt{x + 3} \) at \( a = 1 \) and use it to approximate the numbers \( \sqrt{3.98} \) and \( \sqrt{4.05} \). Are these approximations overestimates or underestimates?

SOLUTION

\[
f'(x) = \frac{1}{2} (x + 3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x + 3}}.\]

\[
L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}.
\]

\[
\sqrt{x + 3} \approx \frac{7}{4} + \frac{x}{4} \quad \text{(when x is near 1)}
\]
\[
\sqrt{3.98} = \sqrt{0.98 + 3} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995
\]
\[
\Rightarrow \sqrt{3.98} < 1.995
\]

and
\[
\sqrt{4.05} = \sqrt{1.05 + 3} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125
\]
\[
\Rightarrow \sqrt{4.05} < 2.0125
\]

Our approximates are overestimates.
**Definition 3.2**

The 1-st order differential $dy$ of $y = f(x)$ at $a$ is defined in terms of $dx$ by equation

\[ df(a) = f'(a) \, dx. \tag{11} \]

**Example 3.2**

If $f(x) = \frac{e^x}{x^2}$, then find $df(1)$

**SOLUTION**

\[
\begin{align*}
  f'(x) &= \frac{e^x \cdot x^2 - e^x \cdot 2x}{x^4} \\
        &= \frac{e^x (x-2)}{x^3} \\
  \Rightarrow f'(1) &= -e.
\end{align*}
\]

So $df(1) = f'(1) \, dx = -edx$. 
Therefore $dy$ represents the amount that the tangent line rises or falls (the change in the linearization), whereas $\Delta y = f(x + \Delta x) - f(x)$ represents the amount that the curve $y = f(x)$ rises or falls when $x$ changes by an amount $dx = \Delta x$. 
**Example 3.3**

Compare the values of $\Delta y$ and $dy$ if 

$$y = f(x) = x^3 + x^2 - 2x + 1$$ 

and $x$ changes 

1. from $2$ to $2.05$ 
2. from $2$ to $2.01$

**Solution** 

$$dy = f'(x)dx = (3x^2 + 2x - 2)dx.$$ 

1. 

$$f(2) = 9, f(2.05) = 9.717625 \Rightarrow \Delta y = 0.717625.$$ 

When $x = 2$ and $dx = \Delta x = 2.05 - 2 = 0.05$, this becomes 

$$dy = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

2. 

$$f(2) = 9, f(2.01) = 9.140701 \Rightarrow \Delta y = 0.140701.$$ 

When $x = 2$ and $dx = \Delta x = 2.01 - 2 = 0.01$, this becomes 

$$dy = [3(2)^2 + 2(2) - 2]0.01 = 0.14$$

$\Delta y \approx dy$ becomes better as $\Delta x$ becomes smaller.
**Definition 3.3**

The 2-nd order differential of \( y = f(x) \) at \( a \) is defined in terms of \( dx \) by equation

\[
d^2f(a) = f''(a)\,dx^2.
\]  

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**Example 3.4**

If \( f(x) = \sqrt{x^2 - 4x + 3} \), then find \( d^2f(0) \)

**Solution**

\[
f''(x) = \frac{-1}{(x^2 - 4x + 3)\sqrt{x^2 - 4x + 3}}
\]

\[\Rightarrow f''(0) = -\frac{1}{3\sqrt{3}}. \text{ So } d^2f(0) = -\frac{1}{3\sqrt{3}}dx^2.\]
**Definition 3.4**

The $n$–th order differential of $y = f(x)$ at $a$ is defined in terms of $dx$ by equation

$$d^n f(a) = f^{(n)}(a) dx^n.$$  

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Rates of change and Related rates

Rates of change in the natural and social sciences

**RATES OF CHANGE**

![Graph showing rates of change](image)

- $m_{PQ} = \text{average rate of change}$
- $m = f'(x_1) = \text{instantaneous rate of change}$

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RATES OF CHANGE

If $x$ changes from $x_1$ to $x_2$, then the change in $x$ is $\Delta x = x_2 - x_1$ and the corresponding change in $y$ is $\Delta y = f(x_2) - f(x_1)$. The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the average rate of change of $y$ with respect to $x$ over the interval $[x_1, x_2]$.

The instantaneous rate of change of $y$ with respect to $x$ or the slope of the tangent line at $P(x_1, f(x_1))$ is

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
If \( s = f(t) \) is the position function of a particle that is moving in a straight line, then

1. \( \frac{\Delta s}{\Delta t} \) represents the **average velocity** over a time period \( \Delta t \)

2. \( \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \) represents the **instantaneous velocity**

3. The instantaneous rate of change of velocity with respect to time is **acceleration**: \( a(t) = \nu'(t) = s''(t) \).
**Example 4.1**

The position of a particle is given by the equation

\[ s = f(t) = t^3 - 6t^2 + 9t, \]

where \( t \) is measured in seconds and \( s \) in meters.

1. **Find the velocity at time \( t \).** What is the velocity after 2s? When is the particle at rest? When is the particle moving forward (that is, in the positive direction) and backward?

2. **Find the total distance travelled by the particle during the first five seconds.**

3. **Find the acceleration at time \( t \) and after 4s.** When is the particle speeding up, slowing down?
The velocity function

\[ v(t) = s'(t) = 3t^2 - 12t + 9 \Rightarrow v(2) = -3 \text{ m/s}. \]

The particle is at rest when

\[ v(t) = 0 \iff 3t^2 - 12t + 9 = 0 \iff \begin{cases} t = 1 \text{ s} \\ t = 3 \text{ s} \end{cases}. \]

The particle moves forward when

\[ v(t) > 0 \iff \begin{cases} t > 3 \\ t < 1 \end{cases}. \]

It moves backward then \( 1 < t < 3 \).
We need to calculate the distances travelled by the particle during the time intervals $[0, 1], [1, 3]$ and $[3, 5]$ separately.

\[
|f(1) - f(0)| + |f(3) - f(1)| + |f(5) - f(3)| =
\]

\[
= |4 - 0| + |0 - 4| + |20 - 0| = 4 + 4 + 20 = 28 \text{m}.
\]
The acceleration is the derivative of the velocity function:

\[ a(t) = v'(t) = s''(t) = 6t - 12 \Rightarrow a(4) = 12 \text{ m/s}^2 \]

The particle speeds up when the velocity is positive and increasing (it means \( v(t) \) and \( a(t) \) are both positive) and also when the velocity is negative and decreasing (it means \( v(t) \) and \( a(t) \) are both negative). In other words, the particle speeds up when the velocity and acceleration have the same sign.

\[ v(t) \cdot a(t) > 0 \iff (3t^2 - 12t + 9)(6t - 12) > 0 \]
\[= 18(t - 1)(t - 3)(t - 2) > 0 \iff \begin{cases} 
t > 3 \\
1 < t < 2
\end{cases}\]

The particle slows down when \(v(t)\) and \(a(t)\) have opposite signs \(v(t) a(t) < 0 \iff \begin{cases} 
2 < t < 3 \\
0 < t < 1
\end{cases}\)
**Related rates**

* If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increase are related to each other.
* In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity.
* The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.
**Example 4.2**

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 50 cm

**SOLUTION** Let \( V(t) \) be the volume of the balloon and let \( r(t) \) be its radius. We start by identifying two things

- the **given information**: the rate of increase of the volume of air is 100 cm³/s \( \Rightarrow \frac{dV}{dt} = 100 \text{ cm}^3/\text{s} \)
2. the *unknown*: the rate of increase of the radius when the diameter is 50\(cm\) \(\Rightarrow\) \(\frac{dr}{dt}\) = ? when \(r = 25\text{\(cm\).}

3. Equation that relates \(V(t)\) and \(r(t)\) is \(V = \frac{4}{3}\pi r^3\)

4. Use the Chain Rule to differentiate both sides with respect to time

\[
\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}
\]

\[\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}\]
If we put $r = 25$ and $\frac{dV}{dt} = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi (25)^2} \cdot 100 = \frac{1}{25\pi} \approx 0.0127 \text{cm/s}.$$
**MatLab: Derivatives**

1. **Derivatives:** `diff(f)` or `diff(f,x)`. **Example:** `syms x; diff(x^2 + 2) ⇒ ans = 2*x`

2. **The n-th derivative:** `diff(f,n)` or `diff(f,x,n)`. **Example:** `syms x; diff(exp(x^2 + 1),4) ⇒ ans = 12 * exp(x^2 + 1) + 48 * x^2 * exp(x^2 + 1) + 16 * x^4 * exp(x^2 + 1).`
THANK YOU FOR YOUR ATTENTION