Chapter 10:

Residue Theory
10.1: Singularities:

- **Singular Point** $a$ of $f(z)$: $f(z)$ not analytic at $z = a$ but analytic in its neighborhood.

- **Isolated Singularity** of $f(z)$ if $z = a$ has a neighborhood without further singularities of $f(z)$.
Isolated Singularities classified:

- Using Laurent Series at isolated singularity \( z = a \):

\[
\sum_{n=-\infty}^{\infty} a_n (z - a)^n = \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z - a)^m}
\]

- (Analytic part)
- (Principal part)

- If principal part is zero: \( z = a \) is removable singularity.

- If principal part contains \( m \) terms: \( z = a \) is pole of order \( m \).
  
  If \( m = 1 \), \( z = a \) is simple pole.

- If principal part contains infinite terms: \( z = a \) is essential singularity.
### Summary:

| $z = z_0$ | Laurent Series for $0 < |z - z_0| < R$ |
|-----------|--------------------------------------|
| Removable singularity | $a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Pole of order $n$ | $\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$ |
| Simple pole | $\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
| Essential singularity | $\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$ |
10.2: mth-order Zeros and Poles:

- **Z = a is an mth-order zero if:**
  \[ f(z) = (z - a)^m \varphi(z) \]
  \( (\varphi(z): \text{analytic and } \varphi(a) \neq 0) \)

- **Z = a is an mth-order pole if:**
  \[ f(z) = \frac{\varphi(z)}{(z - a)^m} \]
  \( (\varphi(z): \text{analytic and } \varphi(a) \neq 0) \)
10.3: Residue:

- Residue of \( f(z) \) at \( z = a \), written \([\text{Res}f(z); a]\), defined:

\[
\text{Res}\{f(z); a\} = a_{-1} \text{ of Laurent series}
\]

- Example: Expand \( f(z) = \frac{1}{(z-1)^2(z-3)} \) to Laurent Series in neighborhood of \( z = 1 \):

\[
f(z) = \frac{-1/2}{(z-1)^2} + \frac{-1/4}{(z-1)} - \frac{1}{8} - \frac{z-1}{16} - \ldots
\]

- Residue of \( f(z) \) at removable point is always 0.
10.4: Calculate Residue at a Pole:

**a) Residue at a simple pole:**

\[
\text{Res}\{f(z), z_0\} = \lim_{z \to z_0} \left\{ f(z)(z - z_0) \right\}
\]

- If \( f(z) = \frac{P(z)}{Q(z)} \) then:
  \[
  \text{Res}\{f(z), z_0\} = \frac{P(z_0)}{Q'(z_0)}
  \]
b) **Residue at a Pole of order m**:  

\[
\text{Res}\{f(z), z_0\} = \frac{1}{(m-1)!} \lim_{{z \to z_0}} \left\{ \frac{d^{m-1}\left[f(z)(z-z_0)^m \right]}{dz^{m-1}} \right\}
\]
10.5: Residue Theorem:

If \( f(z) \) = analytic on and within \( C \), except at a finite number of singular points \( z_0, z_1, \ldots, z_k \) within \( C \), we have:

\[
\oint_{C} f(z) \, dz = 2\pi j \sum_{k=1}^{n} \text{Res}\{f(z), z_k\}
\]