Chapter 9:

Series Complex Functions
9.1: Sequence:

a) Definition:
Sequence = an ordered set of numbers \( a_1, a_2, a_3, \ldots \) so that \( a_n = f(n) \). We can write \( \{f(n)\} \) or \( \{a_n\} \).

b) The Limit of a sequence:
Suppose the limit = \( L \). Given a positive number \( \varepsilon \), we can find a number \( N \) such that: 
\[
|a_n - L| < \varepsilon \quad \text{for all } n > N.
\]

And we can write: 
\[
\lim_{n \to \infty} \{a_n\} = L
\]

L exist : convergent  \hspace{1cm} L not exist : divergent
c) Sequence of Complex functions:

Complex Sequence = an ordered set of complex numbers \( a_1, a_2, a_3, \ldots \) so that \( a_n = f_n(z) \). We can write \( \{f_n(z)\} \) or \( \{a_n\} \).

If \( |f_n(z) - f(z)| < \varepsilon \) for all \( n > N \) : sequence is convergent.

Example 1: Is \( a_n = \frac{2}{n^2} + j3 \) convergent? Find \( N \)?

\[
\lim_{n \to \infty} \{a_n\} = j3 \quad \Rightarrow \quad \left| \frac{2}{n^2} \right| < \varepsilon \quad \Rightarrow \quad N > \sqrt{\frac{2}{\varepsilon}}
\]
9.2: Series Complex Function:

- **Series** = By summing up the terms of a sequence.
  \[ S_n = a_1(z) + a_2(z) + \ldots + a_n(z) \]

- **Infinite Series**: \[ \sum_{n=1}^{\infty} a_n(z) \]

- **Convergent**: \[ \lim_{n \to \infty} S_n = S(z) \] : Series is convergent

  Otherwise : Series is divergent
Convergent Tests:

i. Comparison Test: \[ \sum_{n=1}^{\infty} |a_n| : \text{convergent} \]

ii. Ratio Test: \[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \]

iii. Weierstrass M-test: \[ |a_n| < M_n \quad \text{and} \quad \sum_{n=1}^{\infty} M_n : \text{convergent} \]
9.3: Power Series:

- Having the form:

\[ \sum_{n=1}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots \]

(a_i: complex constant)

- Convergent:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}(z-a)^{n+1}}{a_n z^n} \right| = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| |z-a| = L |z-a| < 1 \]

\[ |z-a| < \frac{1}{L} = R \]

: converges in a disc, center at a, radius of R.
9.4: Taylor Series:

- Taylor series of \( f(z) \), center at \( z = a \), having the form:

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z-a)^n = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots
\]

- MacLaurin Series = Taylor series with center \( a = 0 \):

\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \ldots
\]

- A Taylor series converges in a disk, center at \( a \), radius \( R \).

(Note: \( R = \text{distance from } a \text{ to nearest isolated singularity of } f(z) \).)
Some Special Series:

\[ e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots \]

\[ \frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \quad (\text{if } |z| < 1) \]

\[ \frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots \quad (\text{if } |z| < 1) \]

\[ \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots \]

\[ \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \]
9.5: Laurent Series:

- If \( z = a \) is a singular point of \( f(z) \) [\( f(z) \) not analytic at \( z = a \)], we cannot expand \( f(z) \) in a power series center at \( z = a \).

Example: Cannot expand \( f(z) = \frac{1}{1-z} \) in a power series at \( z_0 = 1 \).

- Can represent \( f(z) \) in a series that contains both negative and positive integer power of \( (z - a) \) : Laurent series.
Laurent Series:

Laurent series of $f(z)$ for all $z$ near $a$, having the form:

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n = \ldots + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

where: $a_n = \frac{1}{2\pi j} \oint_{C} \frac{f(z)}{(z-a)^{n+1}} dz$ \hspace{1cm} (C: annulus enclosing the point $a$)

The sum includes terms with negative powers: principal part.

$$\frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \ldots$$

The rest is analytic part.
Example 1: Laurent Series

Expand \( f(z) = \frac{\sin z}{z^4} \) in a Laurent series about \( z_0 = 0 \)?

We have: \[
\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots
\]

\[
f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1/3!}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \ldots
\]

(Laurent Series)

(Principal part)  (Analytic part)
Find Laurent Series

Formula: \[ a_n = \frac{1}{2\pi j} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \]

is rarely used in practice.

- Obtain Laurent Series by:
  
  i. Employing a known power series.

  ii. Creation of geometric function.