Determine the response of the following systems to the input signal.

\[
x(n) = \begin{cases} 
|n|, & -3 \leq n \leq 3 \\
0, & \text{otherwise}
\end{cases}
\]

(a) \( y(n) = x(n) \) (identity system)
(b) \( y(n) = x(n - 1) \) (unit delay system)
(b) \( y(n) = x(n + 1) \) (unit advance system)
(d) \( y(n) = \frac{1}{3} [x(n + 1) + x(n) + x(n - 1)] \) (moving average filter)
(c) \( y(n) = \text{median} \{x(n + 1), x(n), x(n - 1)\} \) (median filter)
(f) \[ y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots \text{ (accumulator)} \quad (2.2.3) \]
EXAMPLE 2.2.1 Solution

First, we determine explicitly the sample values of the input signal

\[ x(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots \} \]

Next, we determine the output of each system using its input-output relationship.

(a) In this case the output is exactly the same as the input signal. Such a system is known as the *identity* system.
EXAMPLE 2.2.1 Solution

(b) This system simply delays the input by one

\[ y(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots \} \]

(c) In the case the system "advances" the input one sample into the future.

\[ y(n) = \{ \ldots, 0, 3, 2, 1, 0, 1, 2, 3, 0, \ldots \} \]

(d) The output of this system at any time is

\[ y(n) = \{ \ldots, 0, 1, \frac{5}{3}, 2, 1, \frac{2}{3}, 1, 2, \frac{5}{3}, 1, 0, \ldots \} \]
EXAMPLE 2.2.1 Solution

(e) This system selects as its output at time $n$ the median value of the three input samples $x(n - 1)$, $x(n)$, and $x(n + 1)$

$$y(n) = \{ 0, 3, 3, 3, 2, 1, 2, 3, 3, 3, 0, \ldots \}$$

(f) This system is basically an *accumulator*

$$y(n) = \{ \ldots, 0, 3, 5, 6, 6, 7, 9, 12, 0, \ldots \}$$
EXAMPLE 2.2.4

Determine if the system shown in Fig. 2.2.8 are time invariant or time variant.

(a) This system is described by the input–output equations

\[ y(n) = T [x(n)] = x(n) - x(n - 1) \quad (2.2.15) \]

Now if the input is delayed by \( k \) unit in time and applied to the system

\[ y(n,k) = x(n - k) - x(n - k - 1) \quad (2.2.16) \]

If we delay \( y(n) \) by \( k \) unit in time, we obtain

\[ y(n - k) = x(n - k) - x(n - k - 1) \quad (2.2.17) \]

The system is \textit{time invariant}
EXAMPLE 2.2.4

(b) The input-output equation for this system is

\[ y(n) = T[x(n)] = nx(n) \quad (2.2.18) \]

The response of this system to \( x(n - k) \) is

\[ y(n,k) = nx(n - k) \quad (2.2.19) \]

Now if we delay \( y(n) \) in (2.2.18) by \( k \) units in time, we obtain

\[ y(n - k) = (n - k)x(n - k) = nx(n - k) - kx(n - k) \quad (2.2.20) \]

This system is **time variant**, since \( y(n,k) \neq y(n-k) \).
EXAMPLE 2.2.4

(c) This system is described by the input-output relation

\[ y(n) = T [x(n)] = x(-n) \]  \hspace{1cm} (2.2.21)

The response of this system to \( x(n - k) \) is

\[ y(n,k) = T [x(n - k)] = x(-n - k) \]  \hspace{1cm} (2.2.22)

If we delay the output \( y(n) \), as given by (2.2.21), by \( k \) unit in time, the result will be

\[ y(n - k) = x(-n + k) \]  \hspace{1cm} (2.2.23)

Since \( y(n,k) \neq y(n - k) \), the system is \textit{time variant}. 

EXAMPE 2.2.4

(d) The input-output equation for this system is

\[ y(n) = x(n) \cos \omega_0 n \]  \hspace{1cm} (2.2.24)

The response of this system to \( x(n – k) \) is

\[ y(n,k) = x(n – k) \cos \omega_0 n \]  \hspace{1cm} (2.2.25)

If the expression in (2.2.24) is delayed by \( k \) unit and the result is compared to (2.2.25), it is evident that the system is \textit{time variant}.
EXAMPLE 2.2.5

Determine if the systems described by the following input-output equation are linear or nonlinear

(a) \( y(n) = nx(n) \)  \hspace{1cm}  \text{(b) } y(n) = x(n^2) \\

Solution.

(a) For two input sequence \( x_1(n) \) and \( x_2(n) \), the corresponding output are

\[
\begin{align*}
  y_1(n) &= n x_1(n) \\
  y_2(n) &= n x_2(n)
\end{align*}
\]  \hspace{1cm} (2.2.31)
EXAMPLE 2.2.5

A linear combination of the two input sequences results in the output.

\[ y_3(n) = T [a_1 x_1(n) + a_2 x_2(n)] = n [a_1 x_1(n) + a_2 x_2(n)] \]
\[ = a_1 nx_1(n) + a_2 nx_2(n) \quad (2.2.32) \]

A linear combination of the two outputs in (2.2.31) results in the output

\[ a_1 y_1(n) + a_2 y_2(n) = a_1 nx_1(n) + a_2 nx_2(n) \quad (2.2.33) \]

Since the right-hand sides of (2.2.32) and (2.2.33) are identical, the system is **linear**.
EXAMPLE 2.2.5

(b) we find the response of the system to two separate input signals $x_1(n)$ and $x_2(n)$. The result is

$$y_1(n) = x_1(n^2) \quad y_2(n) = x_2(n^2) \quad (2.2.34)$$

The output of the system to a linear combination of $x_1(n)$ and $x_2(n)$ is

$$y_3(n) = T [a_1x_1(n) + a_2x_2(n)] = a_1x_1(n^2) + a_2x_2(n^2) \quad (2.2.35)$$

A linear combination of the two outputs yields

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n^2) + a_2x_2(n^2) \quad (2.2.36)$$

We conclude that the system is linear.
EXAMPLE 2.2.6

Determine if the systems described by the following input-output equations are causal or noncausal.

(a) $y(n) = x(n) - x(n - 1)$  
(b) $y(n) = \sum_{k=-\infty}^{n} x(k)$

(c) $y(n) = ax(n)$  
(d) $y(n) = x(n) + 3x(n + 4)$

(e) $y(n) = x(n^2)$  
(f) $y(n) = x(2n)$  
(g) $y(n) = x(-n)$
EXAMPLE 2.2.6

Solution. The systems described in parts (a), (b), and (c) are clearly *causal*, since the output depends only on the present and past inputs.

The system in parts (d), (e), and (f) are clearly *noncausal*, since the output depends on future values of the input. The system in (g) is also *noncausal*, as we note by selecting, for example, $n = -1$, which yields $y(-1) = x(1)$. Thus the output at $n = -1$ depends on the input at $n = 1$, which is two units of time into the future.
EXAMPLE 2.3.1

Consider the special case of a finite-duration sequence given as $x(n) = \{ 2, 4, 0, 3 \}$

Resolve the sequence $x(n)$ into a sum of weighted impulse sequences.

**Solution.** Since the sequence $x(n)$ is nonzero for the time instants $n = -1, 0, 2$, we need three impulses at delays $k = -1, 0, 2$

Following (2.3.10) we find that

$$x(n) = 2\delta(n +1) + 4\delta(n) + 3\delta(n - 2)$$
EXAMPLE 2.3.3

Determine the output \( y(n) \) of a relaxed linear time-invariant system with impulse response

\[ h(n) = a^n u(n), \ |a| < 1 \]

when the input is a unit step sequence, that is,

\[ x(n) = u(n) \]

Solution

In this case both \( h(n) \) and \( x(n) \) are infinite-duration sequences. We use the form of the convolution formula given by (2.3.28) in which \( x(k) \) is folded.
EXAMPLE 2.3.3 Solution

The sequences \( h(k) \), \( x(k) \), and \( x(-k) \) are shown in Fig.2.24.

The product sequences \( v_0(k) \), \( v_1(k) \), and \( v_2(k) \) corresponding to \( x(-k)h(k) \), \( x(1-k)h(k) \), and \( x(2-k)h(k) \) are illustrated in Fig.2.24(c), (d), and (e), respectively.

Thus we obtain the outputs

\[
\begin{align*}
y(0) &= 1 \\
y(1) &= 1 + a \\
y(2) &= 1 + a + a^2
\end{align*}
\]
EXAMPLE 2.3.3 Solution

Clearly, for \( n > 0 \), the output is

\[
y(n) = 1 + a + a^2 + \ldots + a^n
\]

for \( n < 0 \), the product sequences consist of all zeros. Hence \( y(n) = 0, \, n < 0 \)

A graph of the output \( y(n) \) is illustrated in Fig. 2.3.3(f) for the case \( 0 < a < 1 \). Note the exponential rise in the output as a function of \( n \). Since \(|a| < 1\), the final value of the output as \( n \) approaches infinity is

\[
y(\infty) = \lim_{n \to \infty} y(n) = \frac{1}{1 - a}
\]
EXAMPLE 2.3.3 Solution

Figure 2.24 Graphical computation of convolution in Example 2.3.3

(a) $h(k)$

(b) $x(k)$

(c) $x(-k)$
EXAMPLE 2.3.3 Solution

\[ x(1 - k) \]

\[ x(2 - k) \]

\[ y(n) \]

\[ v_1(k) \]

\[ v_2(k) \]

\[ y(n) \text{ asymptote} \]

\[ a + 1 + a + a^2 \]
EXAMPLE 2.3.4

Determine the impulse response for the cascade of two linear time-invariant systems having impulse responses

\[ h_1(n) = \left(\frac{1}{2}\right)^n u(n) \]

and

\[ h_2(n) = \left(\frac{1}{4}\right)^n u(n) \]

Solution. To determine the overall impulse response of the two systems in cascade, we simply convolve \( h_1(n) \) with \( h_2(n) \).
EXAMPLE 2.3.4 Solution

Hence

\[ h(n) = \sum_{k=-\infty}^{\infty} h_1(k)h_2(n-k) \]

where \( h_2(n) \) is folded and shifted. We define the product sequence

\[ v_n = h_1(k)h_2(n-k) = \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} \]

which is nonzero for \( k \geq 0 \) and \( n - k \geq 0 \) or \( n \geq k \geq 0 \). On the other hand, for \( n < 0 \), we have \( v_n(k) = 0 \) for all \( k \), and hence

\[ h(n) = 0, \ n < 0 \]
EXAMPLE 2.3.4 Solution

For $n \geq k \geq 0$, the sum of the values of the product sequence $v_n(k)$ over all $k$ yields

$$h(n) = \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k \left( \frac{1}{4} \right)^{n-k} = \left( \frac{1}{4} \right)^n \sum_{k=0}^{n} 2^k$$

$$h(n) = \left( \frac{1}{4} \right)^n \left( 2^{n+1} - 1 \right) = \left( \frac{1}{2} \right)^n \left[ 2 - \left( \frac{1}{2} \right)^n \right], \quad n \geq 0$$
EXAMPLE 2.4.4

Determine the homogeneous solution of the system described by the first-order difference equation

\[ y(n) + a_1 y(n - 1) = x(n) \quad (2.4.18) \]

**Solution.** The assumed solution obtained by setting \( x(n) = 0 \) is \( y_h(n) = \lambda^n \)

When we substitute this solution in (2.4.18), we obtain [with \( x(n) = 0 \)]

\[ \lambda^n + a_1 \lambda^{n-1} = 0 \quad \leftrightarrow \quad \lambda = -a_1 \]

The solution to the homogeneous difference equation is \( y_h(n) = C\lambda^n = C (-a_1)^n \quad (2.4.19) \)
EXAMPLE 2.4.4 Solution

The zero-input response of the system can be determined from (2.4.18) and (2.4.19).

With \( x(n) = 0 \), (2.4.18) yields

\[ y(0) = -a_1 y(-1) \]

And from (2.4.19) we have

\[ y_h(0) = C \]

Hence the zero-input response of the system is

\[ y_{zi}(n) = (-a_1)^{n+1} y(-1), \quad n \geq 0 \quad (2.4.20) \]

With \( a = -a_1 \), this result is consistent with (2.4.11) for the first-order system, which was obtained by iteration of the difference equation.
EXAMPLE 2.4.5

Determine the zero-input response of the system described by the homogeneous second-order difference equation

\[ y(n) - 3y(n - 1) - 4y(n - 2) = 0 \quad (2.4.21) \]

Solution. First we determine the solution to the homogeneous equation. We assume the solution to be the exponential

\[ y_h(n) = \lambda^n \]

Upon substitution of this solution into (2.4.21), we obtain the characteristic equation

\[ \lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0 \]
EXAMPLE 2.4.5 Solution

\[ \lambda^{n-2} - (\lambda^2 - 3\lambda - 4) = 0 \]

Therefore, the roots are \( \lambda = -1, 4 \), and the general form of the solution to the homogeneous equation is

\[ y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n \]

\[ = c_1 (-1)^n + c_2 (4)^n \]  \hspace{1cm} (2.4.22)

The zero-input response of the system can be obtained from the homogenous solution by evaluating the constants in (2.4.22), given the initial conditions \( y(-1) \) and \( y(-2) \). From the difference equation in (2.4.21) we have

\[ y(0) = 3y(-1) + 4y(-2) \]
EXAMPLE 2.4.5 Solution

\[ y(1) = 3y(0) + 4y(-1) \]
\[ = 3 [3y(-1) + 4y(-2)] + 4y(-1) \]
\[ = 13y(-1) + 12y(-2) \]

On the other hand, from (2.4.22) we obtain

\[ y(0) = C_1 + C_2 \quad y(1) = -C_1 + 4C_2 \]

By equating these two sets of relations is

\[ C_1 + C_2 = 3y(-1) + 4y(-2) \]
\[ -C_1 + 4C_2 = 13y(-1) + 12y(-2) \]

The solution of these two equations is

\[ C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2) \]
\[ C_2 = \frac{16}{5}y(-1) + \frac{15}{5}y(-2) \]
Therefore, the zero-input response of the system is

\[
y_{zi}(n) = \left[ -\frac{1}{5} y(-1) + \frac{4}{5} y(-2) \right] (-1)^n + \left[ \frac{16}{5} y(-1) + \frac{16}{5} y(-2) \right] (4)^n, \quad n \geq 0
\] (2.4.23)

For example, if \( y(-2) = 0 \) and \( y(-1) = 5 \), then \( C_1 = 1, C_2 = 16 \), and hence

\[
y_{zi}(n) = (-1)^{n+1} + (4)^{n+2}, \quad n \geq 0
\]
EXAMPLE 2.4.6

Determine the particular solution of the first-order difference equation

\[ y(n) + a_1 y(n - 1) = x(n), \quad |a_1| < 1 \quad (1.4.26) \]

when the input \( x(n) \) is a unit step sequence, that is, \( x(n) = u(n) \)

**Solution.** Since the input sequence \( x(n) \) is a constant for \( n \geq 0 \), hence the assumed solution of the difference equation to the forcing function \( x(n) \), called the particular solution of the difference equation, is \( y_p(n) = K u(n) \)
EXAMPLE 2.4.6 Solution

where $K$ is a scale factor determined so that (2.4.26) is satisfied. Upon substitution of this assumed solution into (2.4.26), we obtain

$$K u(n) + a_1 K u(n - 1) = u(n)$$

To determine $K$, we must evaluate this equation for any $n \geq 1$, where none of the terms vanish. Thus

$$K + a_1 K = 1$$

$$K = \frac{1}{1 + a_1}$$
EXAMPLE 2.4.8

Determine the total solution $y(n)$, $n \geq 0$, to the difference equation

$$y(n) + a_1 y(n - 1) = x(n) \quad (2.4.28)$$

When $x(n)$ is a unit step sequence [i.e., $x(n)=u(n)$] and $y(-1)$ is the initial condition

**Solution.** From (2.4.19) of Example 2.4.4, the homogeneous solution is $y_h(n) = C (- a_1)^n$ and from (2.4.26) of Example 2.4.6, the particular solution is

$$y_p(n) = \frac{1}{1 + a_1}$$
EXAMPLE 2.4.8 Solution

Consequently, the total solution is

\[ y(n) = C(-a_1)^n + \frac{1}{1 + a_1} \quad n \geq 0 \quad (2.4.29) \]

Where the constant \( C \) is determined to satisfy the initial condition \( y(-1) \).

We wish to obtain the zero-state response of the system described by the first-order difference equation in (2.4.8). Then we set \( y(-1) = 0 \). To evaluate \( C \), we evaluate (2.4.28) at \( n=0 \) obtaining

\[ y(0) + a_1 y(-1) = 1 \quad \leftrightarrow \quad y(0) = 1 \]
EXAMPLE 2.4.8 Solution

On the other hand, (2.4.29) evaluated at $n = 0$ yields

$$y(0) = C + \frac{1}{1+a_1}$$

Consequently

$$C + \frac{1}{1+a_1} = 1 \iff C = \frac{a_1}{1+a_1}$$

Substitution for $C$ into (2.4.29) yields the zero-state response of the system

$$y_{zs}(n) = \frac{1 - (-a_1)^{n+1}}{1+a_1} \quad n \geq 0$$
EXAMPLE 2.4.8 Solution

If we evaluate the parameter C in (2.4.29) under the condition that $y(-1) \neq 0$, the total solution will include the zero-input response as well as the zero-state response of the system.

In this case (2.4.8) yields

$$y(0) + a_1 y(-1) = 1 \iff y(0) = -a_1y(-1) + 1$$

On the other hand, (2.4.29) yields

$$y(0) = C + \frac{1}{1 + a_1}$$
EXAMPLE 2.4.8 Solution

By equating these two relations, we obtain

\[ C + \frac{1}{1 + a_1} = -a_1 y(-1) + 1 \quad \leftrightarrow \quad C = -a_1 y(-1) + \frac{a_1}{1 + a_1} \]

Finally, if we substitute this value of \( C \) into (2.4.29), we obtain

\[
y(n) = (-a_1)^{n+1} y(-1) + \frac{1 - (-a_1)^{n+1}}{1 + a_1} \quad n \geq 0
\]

\[ = y_{zi}(n) + y_{ze}(n) \]
EXAMPLE 2.6.1

Determine the crosscorrelation sequence $r_{xy}(l)$ of the sequences

$x(n) = \{\ldots, 0, 0, 2, -1, 3, 7, 1, 2, -3, 0, 0, \ldots \}$

$y(n) = \{\ldots, 0, 0, 1, -1, 2, -2, 4, 1, -2, 5, 0, 0, \ldots \}$

Solution. Let us use the definition in (2.6.3) to compute $r_{xy}(l)$. For $l = 0$ we have

$$r_{xy}(0) = \sum_{n=-\infty}^{\infty} x(n) y(n)$$
EXAMPLE 2.6.1 Solution

The product sequence $v_0(n) = x(n)y(n)$ is

$$v_0(n) = \{..., 0, 0, 2, 1, 6, -14, 4, 2, 6, 0, 0, ... \}$$

and hence the sum over all values of $n$ is $r_{xy}(0) = 7$

For $l > 0$, we simply shift $y(n)$ to the right relative to $x(n)$ by $l$ units, compute the product sequence $v_1(n) = x(n) y(n - 1)$, and finally, sum over all values of the product sequence. Thus we obtain

$$r_{xy}(1) = 13, \quad r_{xy}(2) = -18, \quad r_{xy}(3) = 16, \quad r_{xy}(4) = -7,$$

$$r_{xy}(5) = 5, \quad r_{xy}(6) = -3, \quad r_{xy}(l) = 0, \quad l \geq 7$$
EXAMPLE 2.6.1 Solution

For \( l < 0 \), we shift \( y(n) \) to the left relative to \( x(n) \) by \( l \) units, compute the product sequence \( v_1(n) = x(n) y(n - l) \), and sum over all values of the product sequence. Thus we obtain the values of the crosscorrelation sequence

\[
\begin{align*}
  r_{xy}(-1) &= 0, \\
  r_{xy}(-2) &= 33, \\
  r_{xy}(-3) &= -14, \\
  r_{xy}(-4) &= 36, \\
  r_{xy}(-5) &= 19, \\
  r_{xy}(-6) &= -9, \\
  r_{xy}(-7) &= 10, \\
  r_{xy}(l) &= 0, \quad l \leq -8
\end{align*}
\]

Therefore, the crosscorrelation sequence of \( x(n) \) and \( y(n) \) is

\[
r_{xy}(l) = \{ 10, -9, 19, 36, -14, 33, 0, 7, 13, -18, 16, -7, 5, -3 \}
\]