The **z-transform** of the discrete-time system $x(n)$ is defined as the power series

$$x(z) \equiv \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (3.1.1)$$

Where $z$ - complex variable.

It sometimes called **the direct z-transform**.

The inverse procedure is called the **inverse z-transform**.

$$X(z) \equiv Z \{x(n)\} \quad (3.1.2)$$

$$x(n) \leftrightarrow X(z) \quad (3.1.3)$$

The **region of convergence** (ROC) of $X(z)$ is the set of all values $z$ for which $X(z)$ attains a finite value.
3.1 The z-transform

Let us express the complex variable \( z \) in polar form as

\[ z = r \, e^{j \theta} \quad \text{(3.1.4)} \]

\( r = |z| \) and \( \theta = \triangle z \), Then

\[ x(z)|_{z=r e^{j \theta}} = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j \theta n} \]

In the ROC of \( X(z) \), \( |x(z)| < \infty \), then

\[ |X(z)| \leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \quad \text{(3.1.5)} \]

\[ |X(z)| \leq \sum_{n=1}^{\infty} |x(-n) r^{n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^{n}} \right| \quad \text{(3.1.6)} \]
3.1.1 The direct z-transform

Figure 3.1 Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

*ROC* for the first sum consists of all points in a circle of some radius $r_1 < \infty$. 

\[
\sum_{n=1}^{\infty} |x(-n)r^n|
\]
3.1.1 The direct $z$-transform

**ROC** for the second sum consists of all points outside a circle of radius $r > r_2$. 

Region of convergence for 

$$\sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$
3.1.1 The direct z-transform

**ROC** of $X(z)$ is generally specified as the annular region in the $z$-plane, $r_2 < r < r_1$,

$$|X(z)| \quad r_2 < r < r_1$$
3.1.1 The direct z-transform

A discrete–time $x(n)$ is **uniquely** determined by its z-transform $x(z)$ and the region of convergence of $x(z)$.

Table 3.1 Characteristic Families of signal with their corresponding ROC.

<table>
<thead>
<tr>
<th>Signal</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Finite-Duration Signal</strong></td>
<td></td>
</tr>
<tr>
<td>Causal</td>
<td>Entire z-plane except $z=0$</td>
</tr>
</tbody>
</table>
3.1.1 The direct $z$-transform

Anticausal

Entire $z$-plane except $z = \infty$

Two-sided

Entire $z$-plane except $z = 0$ and $z = \infty$
3.1.1 The direct z-transform

Infinite – Duration Signals

Causal

\[ |z| > r_2 \]

Anticausal

\[ |z| < r_1 \]

Two-sided

\[ r_2 < |z| < r_1 \]
3.1.1 The direct $z$-transform

These types of signal are called right-sided, left-sided, and finite-duration two-sided, signals.

If there is a ROC for an infinite duration two-sided signal, it is a ring (annular region) in the $z$-plane.

The one-sided or unilateral $z$-transform given by

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (3.1.11)$$
3.1.2 The Inverse z-Transform

The procedure for transform from the z-domain to the time domain is called the inversion z-transform.

Cauchy integral theorem.

We have

$$X(z) = \sum_{k=-\infty}^{\infty} x(k)z^{-k} \quad (3.1.12)$$

then

$$\int_c X(z)z^{n-1}dz = \int_c \sum_{k=-\infty}^{\infty} x(k)z^{n-1-k}dz \quad (3.1.13)$$

Where $C$ the closed contour in the ROC of $X(z)$. 
3.1.2 The Inverse z-Transform

or

\[ x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} \, dz \quad (3.1.16) \]

**Figure 3.1.5** Contour C for integral in (3.1.13)
3.2 Properties of the z–Transform

+ **Linearity**
  
  if \( x_1(n) \leftrightarrow X_1(z) \) and \( x_2(n) \leftrightarrow X_2(z) \)
  
  then
  
  \[
  x(n) = a_1 x_1(n) + a_2 x_2(n) \quad \leftrightarrow \quad X(z) = a_1 X_1(z) + a_2 X_2(z)
  \]

\[
(3.2.1)
\]

+ **Time shifting**
  
  if \( x(n) \leftrightarrow X(z) \)
  
  then
  
  \[
  x(n-k) \leftrightarrow z^{-k} X(z)
  \]

\[
(3.2.5)
\]

+ **Scaling in the z-domain**
  
  If \( x(n) \leftrightarrow X(z), \quad \text{ROC: } r_1 < |z| < r_2 \)
  
  then \[
  a^n x(n) \leftrightarrow X(a^{-1}z), \quad \text{ROC: } |a|r_1 < |z| < |a|r_2
  \]

\[
(3.2.9)
\]

for any constant \( a \), real or complex.
3.2 Properties of the z–Transform

+ **Time reversal**

  if \( x(n) \leftrightarrow X(z) \), \( \text{ROC: } r_1 < |z| < r_2 \)

  then \( x(-n) \leftrightarrow X(z^{-1}) \), \( \text{ROC: } 1/r_2 < |z| < 1/r_1 \) (3.2.12)

+ **Differentiation in the z-domain**

  if \( x(n) \leftrightarrow X(z) \)

  then \( nx(n) \leftrightarrow -z \frac{dX(z)}{dz} \) (3.2.14)

+ **Convolution of two sequences**

  if \( x_1(n) \leftrightarrow X_1(z) \), \( x_2(n) \leftrightarrow X_2(z) \),

  then \( x(n) = x_1(n) \ast x_2(n) \leftrightarrow X(z) = X_1(z) X_2(z) \) (3.2.17)
3.2 Properties of the z–Transform

+ Correlation of two sequences

If \( x_1(n) \leftrightarrow X_1(z) \), and \( x_2(n) \leftrightarrow X_2(z) \) then

\[
r_{x1x2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \leftrightarrow R_{x1x2}(z) = X_1(z)X_2(z^{-1})
\]

(3.2.18)

+ Multiplication of two sequences

If \( x_1(n) \leftrightarrow X_1(z) \), \( x_2(n) \leftrightarrow X_2(z) \) then

\[
x(n) = x_1(n)x_2(n) \leftrightarrow X(z) = \frac{1}{2\pi j} \oint_C X_1(v)X_2\left(\frac{z}{v}\right) v^{-1} dv
\]

(3.2.19)

C – closed contour that encloses the origin and lies within the region of convergence common to both \( X_1(v) \) and \( X_2(1/v) \)
3.2 Properties of the z–Transform

+ Parseval’s relation

If \(x_1(n)\) and \(x_2(n)\) are complex-valued sequences, then

\[
\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j} \oint_{c} X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv \quad (3.2.22)
\]

provided that \(r_{1l}r_{2l} < 1 < r_{1u}r_{2u}\), where \(r_{1l} < |z| < r_{1u}\) and \(r_{2l} < |z| < r_{2u}\) are the ROC of \(X_1(z)\) and \(X_2(z)\).

+ The initial value theorem

If \(x(n)\) is causal then

\[
x(0) = \lim_{z \to \infty} X(z) \quad (3.2.23)
\]
# 3.2 Properties of the z–Transform

## Table 3.2 Properties of the z-transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Time Domain</th>
<th>z- Domain</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Notation</td>
<td>$x (n)$</td>
<td>$X (z)$</td>
<td>ROC: $r_2 &lt;</td>
</tr>
<tr>
<td>$x_1 (n)$</td>
<td>$X_1 (z)$</td>
<td></td>
<td>$\text{ROC}_1$</td>
</tr>
<tr>
<td>$x_2 (n)$</td>
<td>$X_2 (z)$</td>
<td></td>
<td>$\text{ROC}_2$</td>
</tr>
<tr>
<td><strong>Linearity</strong></td>
<td>$a_1x_1 (n) + a_2x_2 (n)$</td>
<td>$a_1X_1 (z) + a_2X_2 (z)$</td>
<td>At least intersection of $\text{ROC}_1$ and $\text{ROC}_2$</td>
</tr>
<tr>
<td><strong>Time shifting</strong></td>
<td>$x (n-k)$</td>
<td>$z^{-k}X(z)$</td>
<td>That of $X (z)$, except $z=0$ if $k=0$ and $z = \infty$ if $k &lt; 0$</td>
</tr>
<tr>
<td><strong>Scaling in the z-domain</strong></td>
<td>$a^n x (n)$</td>
<td>$X (a^{-1}z)$</td>
<td>$</td>
</tr>
<tr>
<td><strong>Time reversal</strong></td>
<td>$x (-n)$</td>
<td>$X (z^{-1})$</td>
<td>$1/r_1 &lt;</td>
</tr>
<tr>
<td><strong>Conjugation</strong></td>
<td>$x^* (n)$</td>
<td>$X^* (z^*)$</td>
<td>ROC</td>
</tr>
</tbody>
</table>
### 3.2 Properties of the z–Transform

<table>
<thead>
<tr>
<th>Property</th>
<th>Formula</th>
<th>Includes ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real part</td>
<td>( Re{x(n)} )</td>
<td>( 1/2[X(z) + X^<em>(z^</em>)] )</td>
</tr>
<tr>
<td>Imaginary part</td>
<td>( Im{x(n)} )</td>
<td>( 1/2j[X(z) - X^<em>(z^</em>)] )</td>
</tr>
<tr>
<td>Differentiation in</td>
<td>( nx(n) )</td>
<td>( r_2 &lt;</td>
</tr>
<tr>
<td>Convolution</td>
<td>( x_1(n) * x_2(n) )</td>
<td>( X_1(z) ) ( X_2(z) )</td>
</tr>
<tr>
<td>Correlation</td>
<td>( r_{x_1x_2}(l) = x_1(l) * x_2(-l) )</td>
<td>( R_{x_1x_2}(z) = X_1(z) ) ( X_2(z^{-1}) )</td>
</tr>
<tr>
<td>Initial value theorem</td>
<td>If ( x(n) ) causal</td>
<td>( x(0) = \lim X(z) )</td>
</tr>
<tr>
<td>Multiplication</td>
<td>( x_1(n)x_2(n) )</td>
<td>( \frac{1}{2\pi j} \int_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv )</td>
</tr>
<tr>
<td>Parseval's relation</td>
<td>( \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) )</td>
<td>( \frac{1}{2\pi j} \int_C X_1(v)X_2^<em>\left(\frac{1}{v^</em>}\right)v^{-1}dv )</td>
</tr>
</tbody>
</table>
### Table 3.3 Some common z-transform pairs

<table>
<thead>
<tr>
<th>Signal, $x(n)$</th>
<th>$z$-transform, $x(z)$</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$ $\delta(n)$</td>
<td>$1$</td>
<td>All $z$</td>
</tr>
<tr>
<td>$2$ $u(n)$</td>
<td>$\frac{1}{1-z^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$3$ $a^n u(n)$</td>
<td>$\frac{1}{1-az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$4$ $na^n u(n)$</td>
<td>$\frac{az^{-1}}{(1-az^{-2})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$5$ $-a^n u(-n-1)$</td>
<td>$\frac{1}{1-az^{-1}}$</td>
<td>$</td>
</tr>
<tr>
<td>$6$ $-na^n u(-n-1)$</td>
<td>$\frac{az^{-1}}{(1-az^{-1})^2}$</td>
<td>$</td>
</tr>
<tr>
<td>$7$ $(\cos\omega_0 n) u(n)$</td>
<td>$\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0 + z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>
3.2 Properties of the z–Transform

<table>
<thead>
<tr>
<th>Signal, $x(n)$</th>
<th>z- transform, $x(z)$</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 $(\sin \omega_0 n) u(n)$</td>
<td>$\frac{z^{-1} \sin \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>9 $(a^n \cos \omega_0 n) u(n)$</td>
<td>$\frac{1 - az^{-1} \cos \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
<tr>
<td>10 $(a^n \sin \omega_0 n) u(n)$</td>
<td>$\frac{az^{-1} \sin \omega_0}{1 - 2az^{-1} \cos \omega_0 + a^2 z^{-2}}$</td>
<td>$</td>
</tr>
</tbody>
</table>
3.3 Rational z-transforms

+ Poles and Zeros

The **zeros** of a z-transform $X(z)$ are the values of $z$ for which $X(z) = 0$.

The **pole** of a z-transform are value of $z$ for which $X(z) = \infty$.

+ If $X(z)$ is a rational function, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1z^{-1} + \cdots + b_mz^{-m}}{a_0 + a_1z^{-1} + \cdots + a_Nz^{-N}} = \frac{\sum_{k=0}^{M} b_kz^{-k}}{\sum_{k=0}^{N} a_kz^{-k}} \quad (3.3.1)$$
3.3.1 Poles and Zeros

+ If $a_0 \neq 0$, $b_0 \neq 0$

\[
X(z) = \frac{N(z)}{M(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{(z-p_1)(z-p_2)\ldots(z-p_N)}
\]

\[
X(z) = G z^{N-M} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)} \tag{3.3.2}
\]

Where: $G = \frac{b_0}{a_0}$

+ $X(z)$ has $M$ finite zeros at $z = z_1, z_2, \ldots, z_M$

$N$ finite poles at $z = r_1, r_2, \ldots, r_N$
3.3.1 Poles and Zeros

+ We can represent $X(z)$ graphically by a *pole-zero plot* in the complex plane, which shows the location of *poles* by crosses (x) and the location of *zeros* by circles (o).

+ The z-transform $X(z)$ is a complex function of the complex variable $z = \text{Re}(z) + j\text{Im}(z)$.

\[
X(z) = \frac{z}{z - a} \quad \text{ROC: } |z| > a
\]

**Figure 3.7** Pole-zero plot for the causal exponential signal $x(n) = a^n u(n) \quad a > 0$.
3.3.1 Poles and Zeros

+ $|X(z)|$ is a real and positive function of $z$. Since $z$ represents a point in the complex plane, $|X(z)|$ is a two-dimensional function and describes a “surface”.

For example the z-transform

$$X(z) = \frac{z^{-1} - z^{-2}}{1 - 1.2732z^{-1} + 0.81z^{-2}} \quad (3.3.3)$$

Figure 3.3.4 Graph of $|X(z)|$ for the z-transform in (3.3.3)
3.3.2 Pole location and time-domain behavior for causal signals

+ We consider the relation between the *z-plane location* of a pole pair and the *form of the corresponding signal* in the time domain.

+ The *circle* $|z| = 1$ has a radius of 1, it is called the *unit circle*.

+ For example

\[
x(n) = a^n u(n) \leftrightarrow X(z) = \frac{1}{1 - az^{-1}}, \quad ROC: |z| > |a|
\]

having one *zero* at $z_1 = 0$ and the *pole* at $p_1 = a$ on the real axis, (see fig. 3.7, fig.3.11)
3.3.2 Pole location and time-domain behavior for causal signals

**Figure 3.11** Time-domain behavior of a *single-real pole causal signal* as a function of the location of the pole with respect to the unit circle.
3.3.2 Pole location and time-domain behavior for causal signals

The **signal is decaying** if the **pole is inside the unit circle**, **fixed** if the **pole is on the unit circle**, and **growing** if the **pole is outside the unit circle**.

A causal real signal with a **double real pole** has the form: $X(n) = na^n u(n)$ (see table 3.3)

A double real pole on the unit circle results in an **unbound signal** (see Fig 3.3.6)
3.3.2 Pole location and time-domain behavior for causal signals

Figure 3.12 Time-domain behavior of causal signal corresponding to a double \((m=2)\) real pole, as a function of the pole location.
3.3.2 Pole location and time-domain behavior for causal signals

Figure 3.13 illustrates the case of *a pair of complex – conjugate poles*. This configuration of poles results in an *exponentially weighted sinusoidal signal*. The *amplitude* of the signal is *growing* if $r > 1$, *constant* if $r = 1$ (sinusoidal signals), and *decaying* if $r < 1$.

**Figure 3.13** A pair of complex-conjugate poles corresponds to causal signals with oscillatory behavior.
3.3.2 Pole location and time-domain behavior for causal signals

\[ z - \text{plane} \]

\[ x(n) \quad r=1 \]

\[ 0 \quad 1 \]

\[ z - \text{plane} \]

\[ x(n) \quad r^n \]

\[ 0 \quad 1 \]

\[ \omega_0 \]

\[ r \]

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3.3.2 Pole location and time-domain behavior for causal signals

Fig 3.14 shows the behavior of a causal signal with a double pair of poles on the unit circle.

A signal with a pole near the origin decays more rapidly than one associated with a pole near the unit circle.

Everything we have said about causal signals applies as well to causal LTI systems.

**Figure 3.14** Causal signal corresponding to a double pair of complex–conjugate poles on the unit circle.
3.3.3 The system Function of a Linear Time-Invariant System

\[ Y(z) = H(z)X(z) \quad (3.3.4) \]

- \( Y(z) \) - the z-transform of the output sequence \( y(n) \).
- \( X(z) \) - the z-transform of the input sequence \( x(n) \).
- \( H(z) \) - the z-transform of the unit sample response \( h(n) \).

\[ H(z) = \frac{Y(z)}{X(z)} \quad (3.3.5) \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \quad (3.3.6) \]

\( H(z) \) is called the **systems function**.

We have linear constant-coefficient difference equation:

\[ y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k (n-k) \quad (3.3.7) \]
3.3.3 The system Function of a Linear Time-Invariant System

By applying the time-shifting property, we obtain.

\[
Y(z) = - \sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}
\]

\[
\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{\infty} a_k z^{-k}}
\]

Or

\[
H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}
\]  

(3.3.8)

+ If \( a_k = 0 \) for \( 1 \leq k \leq N \), then

\[
H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}
\]  

(3.3.9)

\( H(z) \) contains \( M \) zeros, \( M \text{th-order} \) at the origin \( z = 0 \)

It is called an \textit{all-zero system}. 
System has a finite-duration impulse response (FIR), and it is called an **FIR system** or **moving average (MA) system**.

+ If \( b_k = 0 \) for \( 1 \leq k \leq M \), then

\[
H(z) = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}} \quad a_0 \equiv 1 \quad (3.3.10)
\]

\( H(z) \) consists of \( N \) pole, and an \( N \) th-order zero at the origin \( z = 0 \). This system is called an **all-pole system**.

The impulse response of such a system is infinite in duration, and hence it is an **IIR system**.

The general form of the system by (3.3.8) is called a **pole-zero system**, with \( N \) poles and \( M \) zeros and in **an IIR system**.
3.4 Inversion of the z-transform

The inverse z-transform is formally given by

\[ x(n) = \frac{1}{2\pi j} \oint_C X(z)z^{n-1}dz \]  

(3.4.1)

C - a circle in the ROC of x(z) in the z-plane

+ **There are 3 methods** for the evaluation

1. **Direct evaluation** of (3.4.1), by contour integration.
2. **Expansion into a series of terms**, in the variables z, and z^{-1}
3. **Partial-fraction expansion** and table lookup.
3.4.1 Inverse z-transform by contour Integration + Cauchy residue theorem

Let \( f(z) \) be a function of the complex variable \( z \), and \( C \) be a closed path in the \( z \)-plane.

If the derivative \( df(z)/dz \) exists on and inside the contour \( C \) and if \( f(z) \) has no poles at \( z = z_0 \), then

\[
\frac{1}{2\pi j} \oint_C \frac{f(z)}{z-z_0} \, dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}
\]

(3.4.2)
3.4.1 Inverse z-transform by contour Integration

If the \((k+1)\) order derivative of \(f(z)\) exists and \(f(z)\) has no poles at \(z = z_0\), then

\[
\frac{1}{2\pi j} \oint_C \frac{f(z)}{(z-z_0)^k} \, dz = \left\{ \begin{array}{l}
\frac{1}{(k-1)!} \frac{d^{k-1}f(z)}{dz^{k-1}} \bigg|_{z=z_0} \quad \text{if } z_0 \text{ is outside } C \quad (3.4.3)
\end{array} \right.
\]

Suppose that

\[
p(z) = \frac{f(z)}{g(z)}
\]

\(f(z)\) has no pole inside the contour \(C\)

\(g(z)\) is a polynomial with distinct roots \(z_1, z_2, \ldots, z_n\) inside \(C\) then

\[
\frac{1}{2\pi j} \oint_C \frac{f(z)}{g(z)} \, dz = \sum_{i=1}^{n} A_i \quad (3.4.4)
\]
Chapter 3: Inverse z-transform by contour Integration

Where \( A_i(z) = (z - z_i)p(z) = (z-z_i)f(z)/g(z) \)
(3.4.5)

\( \{ A_i (z_i) \} \) are residues of the corresponding poles at \( z = z_i, \ i = 1,2,\ldots,n. \)

In the case the inverse z-transform we have

\[
x(n) = \sum_i (z-z_i)X(z)z^{n-1} \bigg|_{z=z_i}
\]

If \( X(z)z^{n-1} \) has no poles inside the contour \( C \) for one or more values of \( n \), then \( x(n) = 0 \) for these values.
3.4.2 Inverse z-transform by Power Series

Expansion

Given a z-transform $X(z)$ with its corresponding ROC, we can expand $X(z)$ into a power series of the form.

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \quad (3.4.7)$$

Which converges in the given ROC. Then, by uniqueness of the z-transform, $X(n) = c_n$ for all $n$. When $X(z)$ is rational, the expansion can be performed by long division.
For example

Determine the inverse $z$-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Where ROC: $|z| > 1$
3.4.2 Inverse z-transform by Power Series Expansion

Solution.

Since the ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus, we seek a power series expansion in negative powers of $z$.

$$x(z) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{31}{16}z^{-4} + \cdots$$

by comparing this relation with (3.11) we conclude that

$$x(n) = \left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots \right\}$$
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

The function $X(z)$ as a linear combination.

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \ldots + \alpha_K X_K(z) \quad (3.4.8)$$

Where $X_1(z), \ldots, X_k(z)$ are expressions with inverse transform $x_1(n), \ldots, x_k(n)$ available in a table of z-transform pairs. Then,

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \ldots + \alpha_K x_K(n) \quad (3.4.9)$$

We assume that $a_0 = 1$, so that (3.3.1) can be expressed as

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 b_1 z^{-1} + \ldots + b_M z^{-M}}{1 + a_1 z^{-1} + \ldots + a_N z^{-N}} \quad (3.4.10)$$
### 3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Any *improper rational function* \((M \geq N)\) can be expressed as.

\[
X(z) = \frac{N(z)}{D(z)} = c_0 c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)} + \frac{N_1(z)}{D_1(z)} \tag{3.4.11}
\]

We perform a partial fraction expansion of the proper rational function. From (3.4.10) with \(a_N \neq 0\) and \(M < N\).

Then, we invert each of the terms.

\[
X(z) = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N} \tag{3.4.13}
\]

\[
\Rightarrow \quad \frac{X(z)}{z} = \frac{b_0 z^{N-1} + b_1 z^{N-2} + \cdots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \cdots + a_N} \tag{3.4.14}
\]
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Distinct poles

Suppose that the poles $p_1, p_2, \ldots p_N$ are all different. Then we seek an expansion of the form

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots + \frac{A_N}{z-p_N} \quad (3.4.15)$$

$$\Leftrightarrow \quad \frac{(z-p_k)X(z)}{z} = \frac{(z-p_k)A_1}{z-p_1} + \ldots + A_k + \ldots + \frac{(z-p_k)A_N}{z-p_N} \quad (3.4.20)$$

with $z = p_k$,

$$A_k = \frac{(z-p_k)X(z)}{z} \bigg|_{z = p_k}, \quad k = 1, 2, \ldots, N \quad (3.4.21)$$
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Multiple-order poles

If $X(z)$ has of multiplicity $l$, that is, it contains in its denominator the factor $(z - p_k)^l$, then the expansion (3.4.15) is no longer true. The partial factor expansion must contain the terms.

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \cdots + \frac{A_{lk}}{(z - p_k)^l}$$
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

Now, first \( X(z) \) contains distinct poles.

\[
X(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}} \tag{3.4.27}
\]

From \( x(n) = Z^{-1}\{X(z)\} \), then

\[
Z^{-1}\left\{ \frac{1}{1 - P_k z^{-1}} \right\} = \begin{cases} 
(p_k)^n u(n), & \text{if } ROC: |z| > |p_k| (causal signals) \\
-(p_k)^2 u(-n-1), & \text{if } ROC: |z| < |p_k| (anticausal signals)
\end{cases} \tag{3.4.28}
\]

With \(|z| > p_{\text{max}}\) where \( p_{\text{max}} = \max \{|p_1|\} \)

Then

\[
x(n) = (A_1 p_1^n + A_2 p_2^n + \cdots + A_N p_N^n) u(n) \tag{3.4.29}
\]

If all poles are \textit{real}, (3.4.29) is a linear combination of real exponential signals.
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

If all poles are distinct but some of then are complex,

\[ x_k(n) = [A_k(p_k)^n + A_k(p_k^*)^n]u(n) \quad (3.4.30) \]

\[ A_k = |A_k|e^{j\alpha_k} \quad (3.4.31) \]

\[ P_k = |A_k|e^{j\beta_k} \quad (3.4.32) \]

\[ x_k(n) = |A_k|r_k^n\left[ e^{j(\beta_k n + \alpha_k)} + e^{-j(\beta_k n + \alpha_k)} \right]u(n) \]

or

\[ x_k(n) = 2|A_k|r_k^n \cos(\beta_k n + \alpha_k)u(n) \quad (3.4.33) \]
3.4.3 The inverse z-Transform by Partial-Fraction Expansion

\[ Z^{-1}\left\{ \frac{A_k}{1-p_kz^{-1}} + \frac{A^*_k}{1-p^*_kz^{-1}} \right\} = 2|A_k|r^n_k \cos(\beta_k n + \alpha_k) u(n) \quad (3.4.34) \]

if the ROC: is \( |z| > |p| = r_k \)

\( X(z) \) has **multiple poles**.

\[ Z^{-1}\left\{ \frac{pz^{-1}}{(1-pz^{-1})^2} \right\} = np^n u(n) \quad (3.4.35) \]

provided that the ROC is \( |z| > |p| \)
3.4.4 Decomposition of Rational z-Transforms

If $X(z)$ expressed as:

$$X(z) = \frac{\sum_{k=1}^{M} (1 - z_k z^{-1})}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \prod_{k=1}^{M} (1 - z_k z^{-1}) \prod_{k=1}^{N} (1 - p_k z^{-1})$$  \hspace{1cm} (3.4.40)

If $M \geq N$, $a_0 \equiv 1$, then

$$X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + X_{pr}(z) \hspace{1cm} (3.4.41)$$

$$X_{pr}(z) = A_1 \frac{1}{1 - p_1 z^{-1}} + A_2 \frac{1}{1 - p_2 z^{-1}} + \cdots + A_N \frac{1}{1 - p_N z^{-1}} \hspace{1cm} (3.4.42)$$

$$\frac{A_1}{1 - p z^{-1}} + \frac{A_1^*}{1 - p^k z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \hspace{1cm} (3.4.43)$$
3.4.4 Decomposition of Rational z-Transforms

Where

\[ b_0 = 2 \text{Re}(A) \quad a_1 = -2 \text{Re}(p) \quad (3.4.44) \]
\[ b_1 = 2 \text{Re}(A_p^*) \quad a_2 = |p|^2 \]

\[ \Rightarrow X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + \sum_{k=1}^{K_1} \frac{b_k}{1 + a_k z^{-1}} + \sum_{k=1}^{K_2} \frac{b_{0k} + b_{1k} z^{-1}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}} \quad (3.4.45) \]

Where \( k_1 + 2k_2 = N \)

Assuming for simplicity that \( M = N \)

\[ X(z) = b_0 \prod_{k=1}^{K_1} \frac{1 + b_k z^{-1}}{1 + a_k z^{-1}} \prod_{k=1}^{K_2} \frac{1 + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 + a_{1k} z^{-1} + a_{2k} z^{-2}} \quad (3.4.48) \]

Where

\[ b_{1k} = -2 \text{Re}(z_k), \quad a_{1k} = -2 \text{Re}(p_k) \]
\[ b_{2k} = |z_k|^2, \quad a_{2k} = |p_k|^2 \quad (3.4.47) \]
3.5 The one-side z-Transform

Definition and properties

The one-sided or unilateral z-transform of a signal $x(n)$ is defined

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (3.5.1)$$

notations $Z^+\{x(n)\}$ and $x(n) \rightarrow X^+(z)$
3.5.1 Definition and properties

The one-side z-transform has the following characteristics:

1. It does not contain information about the signal $x(n)$ for negative values of time.
2. It is unique only for causal signals, because only these signals are zero for $n < 0$
3. The ROC of $X^+(z)$, is always the exterior of a circle.
3.5.1 Definition and properties

**Shifting Property**

**Case 1:** Time delay if

\[ x(n) \xrightarrow{z^+} X^+(z) \]

then

\[ x(n-k) \xrightarrow{z^{-k}} z^{-k}[X^+(z) + \sum_{n=1}^{k} x(-n)z^n] \quad k > 0 \quad (3.5.2) \]

In case \( x(n) \) is causal, then

\[ x(n-k) \xrightarrow{z^+} z^{-k}X^+(z) \quad (3.5.3) \]

\[ Z^+[x(n-k)] = [x(-k)+x(-k+1)z^{-1} + \ldots + x(-1)z^{-k+1}] + z^{-k}X^+(z) \quad k > 0 \quad (3.5.4) \]
3.5.1 Definition and properties

Case 2: Time advance if 

\[ x(n) \xrightarrow{z^+} X^+(z) \]

then

\[ x(n + k) \xrightarrow{z^+} z^k \left[ X^+(z) - \sum_{n=0}^{k-1} x(n) z^{-n} \right] k > 0 \quad (3.5.5) \]

**Final value theorem**

if 

\[ x(n) \xrightarrow{z^+} X^+(z) \]

then

\[ \lim_{n \to \infty} x(n) = \lim_{z \to 1} (z - 1) X^+(z) \quad (3.5.6) \]

The limit in (3.5.6) exists if the ROC of \( (z-1) X^+(z) \) includes the unit circle.
3.5.2 Solution of Difference Equations

By reducing the difference equation relating the two time-domain signals to an equivalent algebraic equation relating their one-sided z-transforms.

This equation can be easily solved to obtain the transform of the desired signal.

The signal in the time domain is obtained by inverting the resulting z-transform.
3.6 Analysis of Linear Time – invariant System in the z-Domain

Response of system with rational system functions

If \( x(n) \) has a rational z-transform \( X(z) \) of the form

\[
X(z) = \frac{N(z)}{Q(z)} \quad (3.6.1)
\]

we represent

If the system is initially relaxed, that is \( y(-1) = y(-2) = \ldots = y(-N) = 0 \), then

\[
Y(z) = H(z)X(z) = \frac{B(z)N(z)}{A(z)Q(z)} \quad (3.6.2)
\]
3.6.1 Response of system with rational system functions

If system contains simple poles \( p_1, p_2, \ldots, p_N \) and z-transform of the input signal contains poles \( q_1, q_2, \ldots, q_L \). Where \( p_k \neq q_m \), then

\[
Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{Q_k}{1 - q_k z^{-1}} \quad (3.6.3)
\]

The inverse transform of \( Y(z) \) yields

\[
y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n) \quad (3.6.4)
\]

Scale factors \( \{A_k\} \) and \( \{Q_k\} \) are functions of both sets of poles \( \{p_k\} \) and \( \{q_k\} \).

The first part called the **natural response** of the system. The second part is called **forced response** of the system.
Suppose that $X(n)$ is applied to the pole-zero system at $n = 0$. ($x(n)$ is causal)

Since the input $x(n)$ is causal and output $y(n)$ for $n \geq 0$

$$Y^+(z) = - \sum_{k=1}^{N} a_k z^{-k} \left[ Y^+(z) + \sum_{n=1}^{k} y(-n) 2^n \right] + \sum_{k=0}^{M} b_k z^{-k} X^+(z) \quad (3.6.5)$$

Or

$$Y^+(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} X(z) - \frac{\sum_{K=1}^{N} a_k z^{-k} \sum_{n=1}^{K} y(-n) z^k}{1 + \sum_{K=1}^{N} a_k z^{-k}} \quad (3.6.6)$$

$$= H(z)X(z) + \frac{N_0(z)}{A(z)}$$
3.6.2 Response of poles – zero system with Nonzero Initial conditions

Where

\[ N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n)z^n \]  

(3.6.7)

\[ Y_{zs}(z) = H(z)X(z) \]  

(3.6.8)

\[ Y_{zi}^+(z) = \frac{N_0(z)}{A(z)} \]  

(3.6.9)

Thus,

\[ y(n) = y_{zs}(n) + y_{zi}(n) \]  

(3.6.10)

Since \( A(z) \) has poles as \( p_1, p_2, \ldots, p_N \), then

\[ y_{zi}(n) = \sum_{k=1}^{N} D_k (p_k)^m u(n) \]  

(3.6.11)
3.6.2 Response of poles – zero system with Nonzero Initial conditions

This can be added to (3.6.4)

\[
y(n) = \sum_{k=1}^{N} A'_{k} (p_{k})^{n} u(n) + \sum_{k=1}^{L} Q_{k} (q_{k})^{n} u(n) \quad (3.6.12)
\]

Where

\[
A'_{k} = A_{k} + D_{k} \quad (3.6.13)
\]

The effect of the initial conditions is to alter the natural response of the system though modifications of the scale factors \{ A_{k} \}

There are no new poles introduced by the nonzero initial conditions.
3.6.3 Transient and Steady – state Responses

The **natural response** of a causal system has the form

$$y_{nr}(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n)$$  \hspace{1cm} (3.6.14)

Where \( \{ p_k \} , \ k = 1, 2 \ldots, N \) are the poles of the system.
\( \{ A_k \} \) are scale factors

The **forced response** of the system has the form

$$y_{fr}(n) = \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$  \hspace{1cm} (3.6.15)

Where \( \{ q_k \} , \ K = 1, 2, \ldots, L \) are the poles
\( \{ Q_k \} \) are scale factors
when the *causal input signal* is a *sinusoid* the poles fall on the unit circle, consequently the *forced response* is also a *sinusoid*. It is called the *steady-state response* of the system.
3.6.4 Causality and Stability

A linear time-invariant system is causal if and only if the ROC of the system function is the exterior of the circle of radius $r < \infty$, including the point $z = \infty$.

A necessary and sufficient conditions for a linear time-invariant system to be **BIBO stable** is

$$
\sum_{n=-\infty}^{\infty} |H(n)| < \infty
$$
3.6.4 Causality and Stability

\[ H(z) \] must contain the unit circle within its ROC.

\[
H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}
\]

when on the unit circle (\(|z| = 1\))

\[
|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|
\]

A linear time-invariant system is BIBO stable if and only if the ROC of the system function includes the unit circle.

A causal linear time-invariant system is BIBO stable if and only if the poles of \( H(z) \) are inside the unit circle.
When a z-transform has a pole that is at the same location as a zero, the pole is canceled by the zero.

Pole-zero cancellations can occur either in the system function itself or in the product of the system function with the z-transform of the input signal.
3.6.6 Multiple–Order poles and stability

The input is bounded if its z-transform contains pole \( \{ q_k \} \), \( k = 1, 2, \ldots, L \) which satisfy the condition \( |q_k| \leq 1 \) for all \( k \).

Thus, the forced response of the systems is also bounded, even when the input signal contains one or more distinct poles on the unit circle.
3.6.8 Stability of Second–Order Systems

A causal two-pole system described by the second-order difference equation.

\[ y(n) = -a_1y(n-1) - a_2y(n-2) + b_0x(n) \]  \hspace{1cm} (3.6.26)

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + a_1z^{-1} + a_2z^{-2}} = \frac{b_0z^2}{z^2 + a_1z + a_2} \]  \hspace{1cm} (3.6.27)

\[ p_1, p_2 = -\frac{a_1}{2} \pm \frac{\sqrt{a_1^2 - 4a_2}}{4} \]  \hspace{1cm} (3.6.28)

For stability

\[ |a_2| = |p_1p_2| = |p_1||p_2| < 1 \]  \hspace{1cm} (3.6.31)

\[ |a_1| < 1 + a_2 \]  \hspace{1cm} (3.6.32)
3.6.8 Stability of Second–Order Systems

Figure 3.15 Region of stability (stability triangle) in the \((a_1, a_2)\) coefficient plane for a second-order system.

Problems : 3.1, 3.2, 3.5, 3.11, 3.15, 3.43, 3.47.